Generalized Central Limit Theorems for strongly correlated variables from a renormalization group perspective

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PhLAM Physique des Lasers Atomes et Molécules

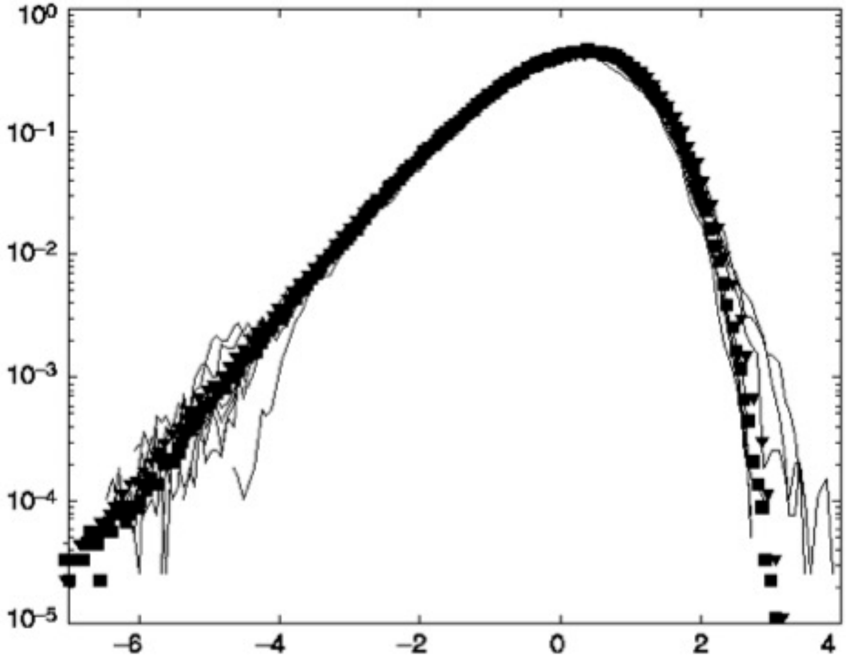
Introduction

Statistics/probability needed for complex systems (many degrees of freedom, correlations, etc.)

Ex: Econometrics/finance, out-of-equilibrium systems (KPZ, disease propagation), statistical physics, …

When d.o.f independent or weak correlations: Central Limit Theorem (CLT) and generalization (Levy distribution)

For strong correlations, no general method…



Fluctuation of the energy in turbulence
Fluctuation of magnetization in 2D spin model
Bramwell, Holdsworth & Pinton ‘98

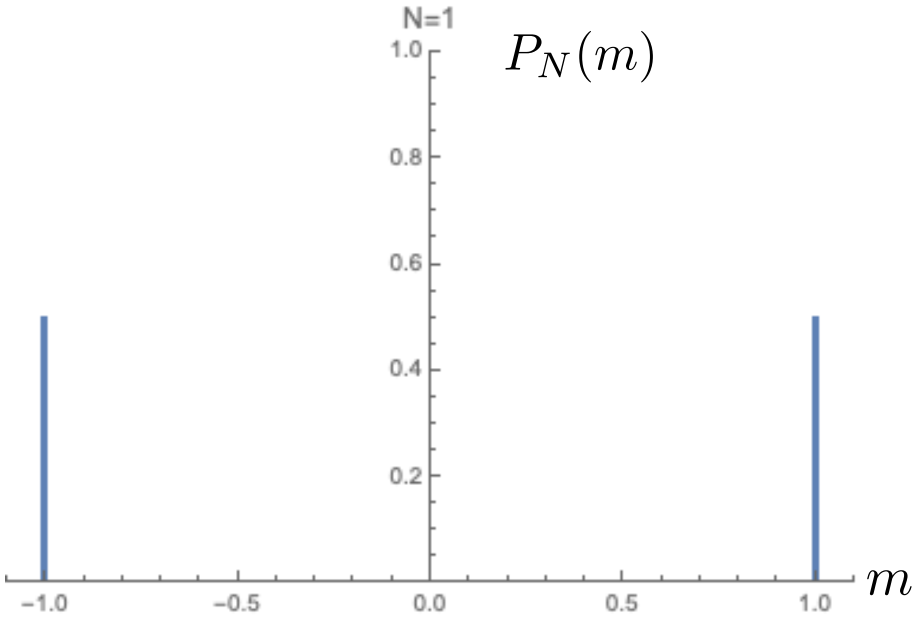
Central Limit Theorem

Take N random i.i.d variables $s_i = \pm 1$ (coin flips, spins, Brownian motion,…)

Distribution of the mean for large N, $m = \frac{1}{N}\sum_i s_i$

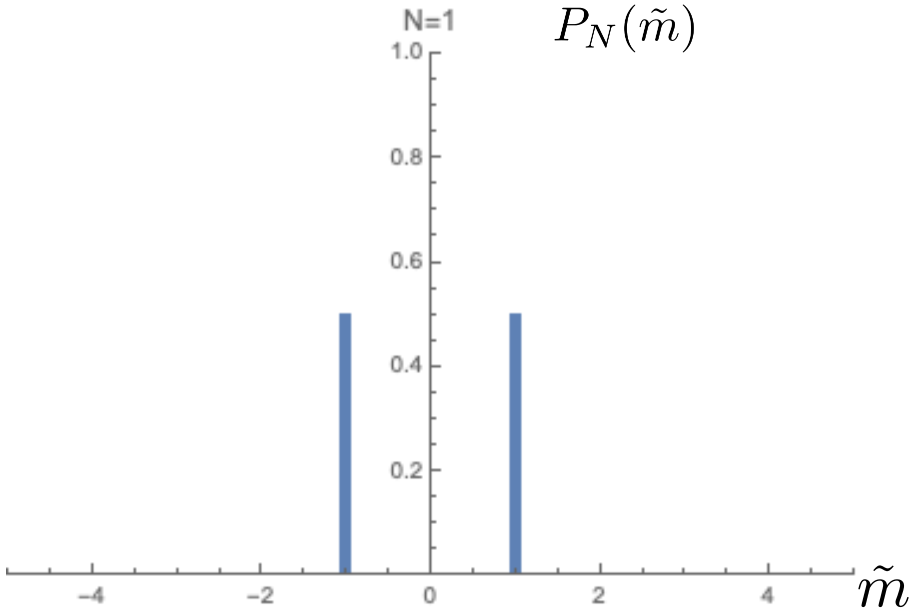
$$\tilde{m} = \frac{1}{\sqrt{N}}\sum_i s_i$$

Law of large number



$$\langle m^2 \rangle = \frac{1}{N}$$

Central Limit Theorem (CLT)

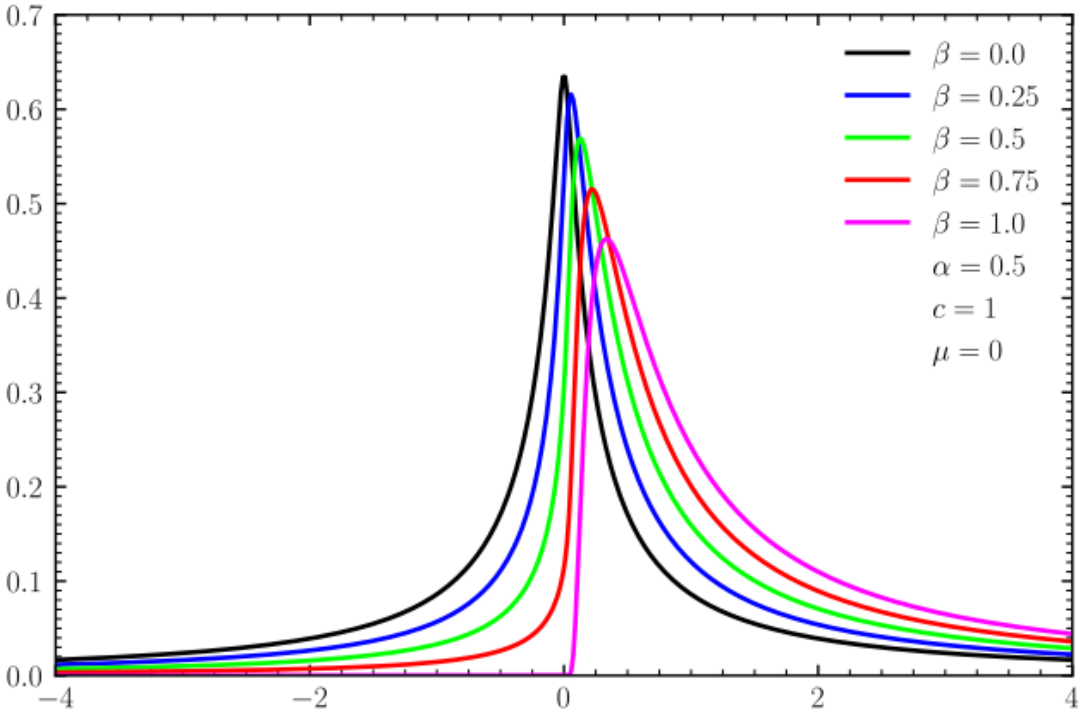


$$\langle \tilde{m}^2 \rangle = 1$$

Stable laws

Stable law: if X and Y are i.i.d.'s with law P, then M=a X+b Y has the same law (e.g. Normal distribution)

Stable laws for i.i.d.'s known
(Levy alpha-stable laws)



Generalized CLT for i.i.d.'s

Sum of i.i.d. random variables (correctly normalized) will converge to such a stable law

Generalized CLT for strongly correlated variables

$$\tilde{m} = \frac{1}{\sqrt{N}} \sum_i s_i$$

Correlations: $\langle s_i s_j \rangle = G(|i-j|)$

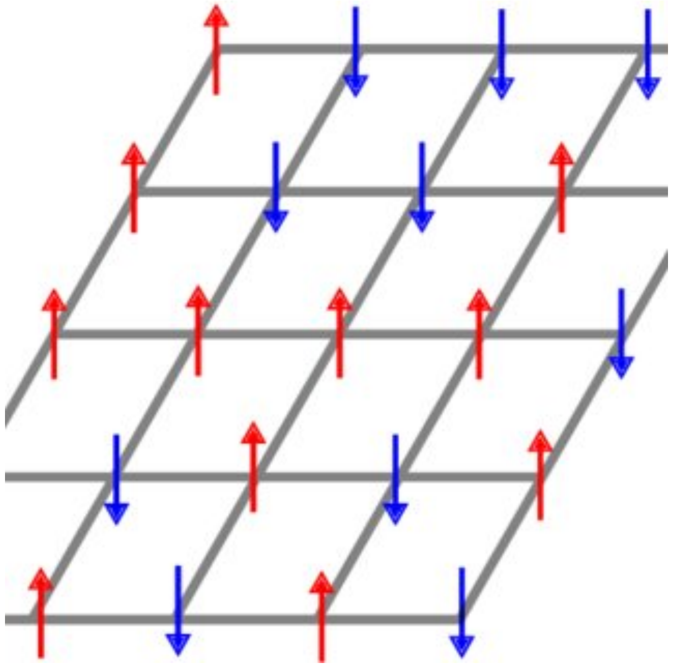
Weak correlations: $\langle \tilde{m}^2 \rangle \propto N^0$ Similar to i.i.d.'s: standard CLT

Strong correlations: $\lim_{N\to\infty} \langle \tilde{m}^2 \rangle = \infty$

Example: Second order phase transitions

Standard model of statistical physics: Ising model

$$P(\{s_i\}) \propto e^{-H/T}$$



Energy of configuration: $H = -J \sum_{\langle i,j \rangle} s_i s_j$

Number of spins: $N = L^d$

Basics of the Ising model

Phase transition from paramagnet at high T ($\langle s_i \rangle = 0$) to a ferromagnet at low T ($\langle s_i \rangle \neq 0$)

High temperature (T>>T_c): finite correlation length $\langle s_i s_j \rangle \propto e^{-|i-j|/\xi}$

Close to T_c, many emergent phenomena:

- scale invariance at T_c $\langle s_i s_j \rangle \propto \frac{1}{|i-j|^{d-2+\eta}}$

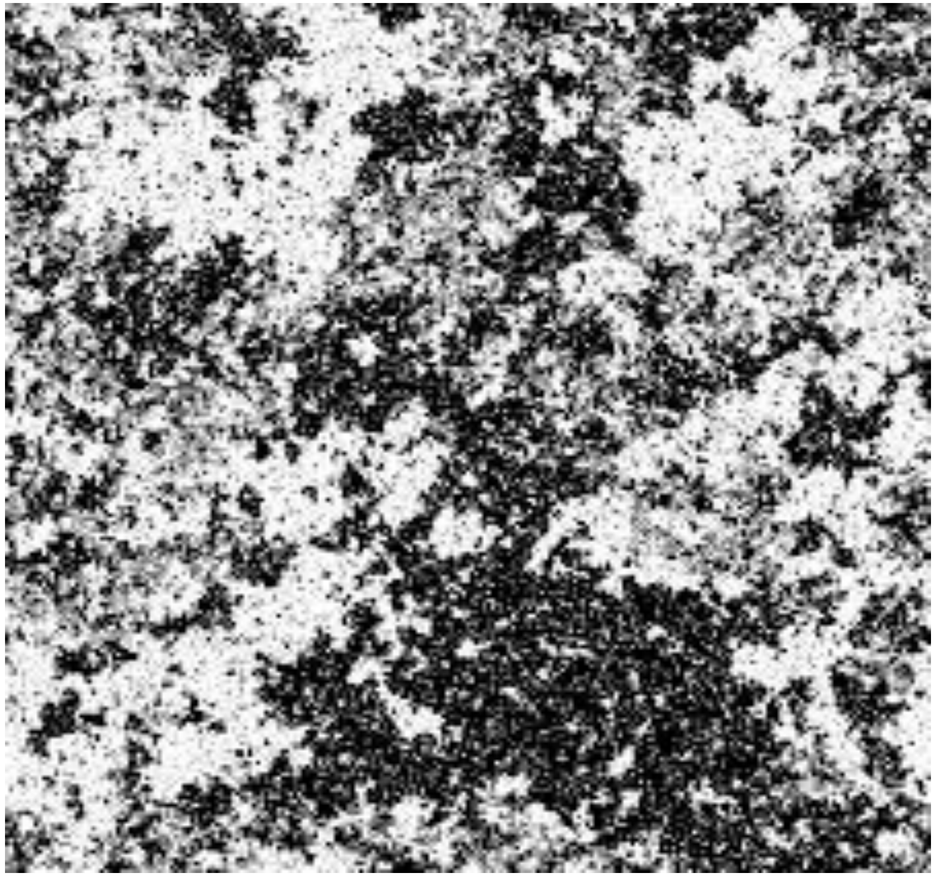
- scaling $\xi \propto |T_c - T|^{-\nu}$
  $\langle s_i \rangle \propto (T_c - T)^{\beta}$
  $\frac{\beta}{\nu} = \frac{d-2+\eta}{2}$

- universality (e.g. liquid-gas critical point, many ferromagnets)

- conformal invariance, fractal dimension of domains,…

$\xi, L \to \infty$

L



Distribution of the magnetization

$$\tilde{m} = \frac{1}{\sqrt{N}} \sum_i s_i$$

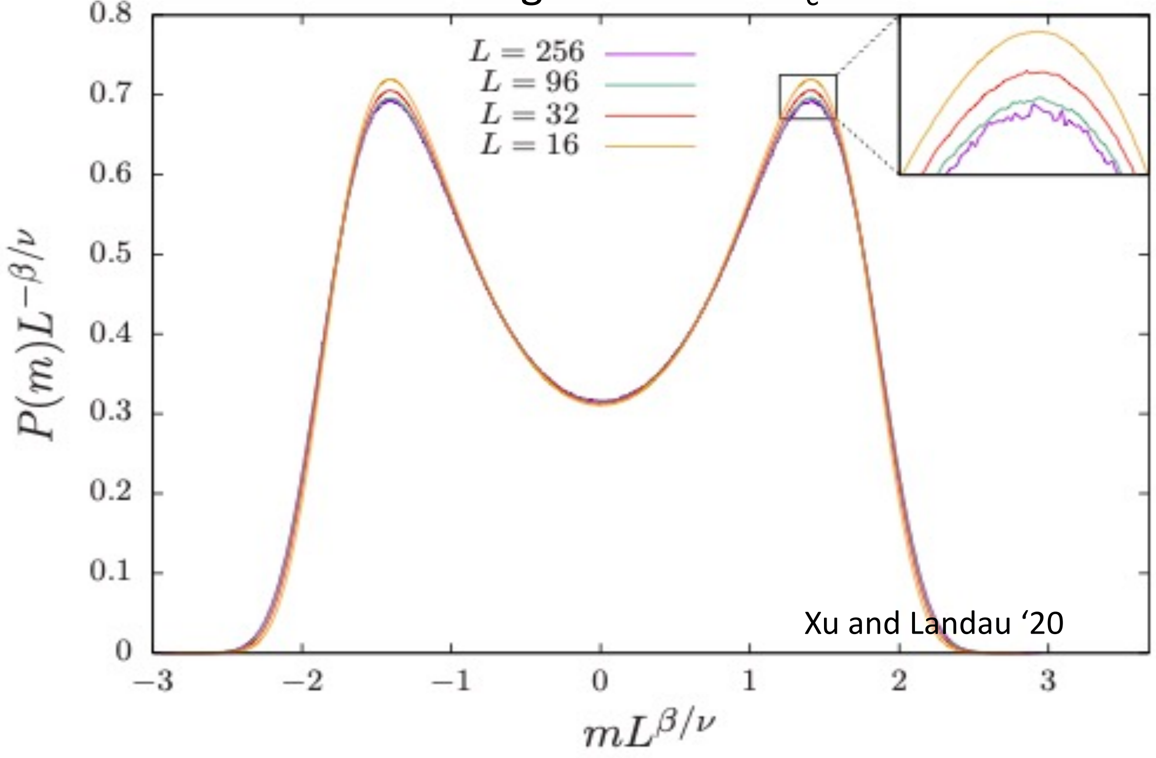
High temperature phase: many independent blocks $\langle \tilde{m}^2 \rangle \propto \xi^2$

Weak correlations and gaussian distribution

Exactly at T_c : $\langle \tilde{m}^2 \rangle \propto L^{2-\eta}$ Strong correlations and non-Gaussian distribution

$$\langle m^2 \rangle \propto L^{-d+2-\eta} = L^{-2\beta/\nu}$$

3D Ising model at $T=T_c$



Stable law
for strongly correlated variables

$$P(m) \propto e^{-I^*(mL^{\beta/\nu})}$$

Understanding the scaling of typical magnetization fluctuations

Bouchaud & Georges ‘90

$$\langle m^2 \rangle \propto L^{-d+2-\eta} = L^{-2\beta/\nu}$$

Number of “truly correlated” spins with a spin at position *i*: $N_{cor} \sim \sum_j \langle s_i s_j \rangle \sim L^{2-\eta}$

Assume that these correlated spins behave as a block-spin $S_I = \sum_{i \in I} s_i$

Number of block spins $N_{BS} \sim \frac{L^d}{N_{cor}} \sim L^{d-2+\eta}$

Total magnetization $m = \frac{1}{L^d} \sum_i s_i = \frac{1}{N_{BS}} \sum_I S_I$

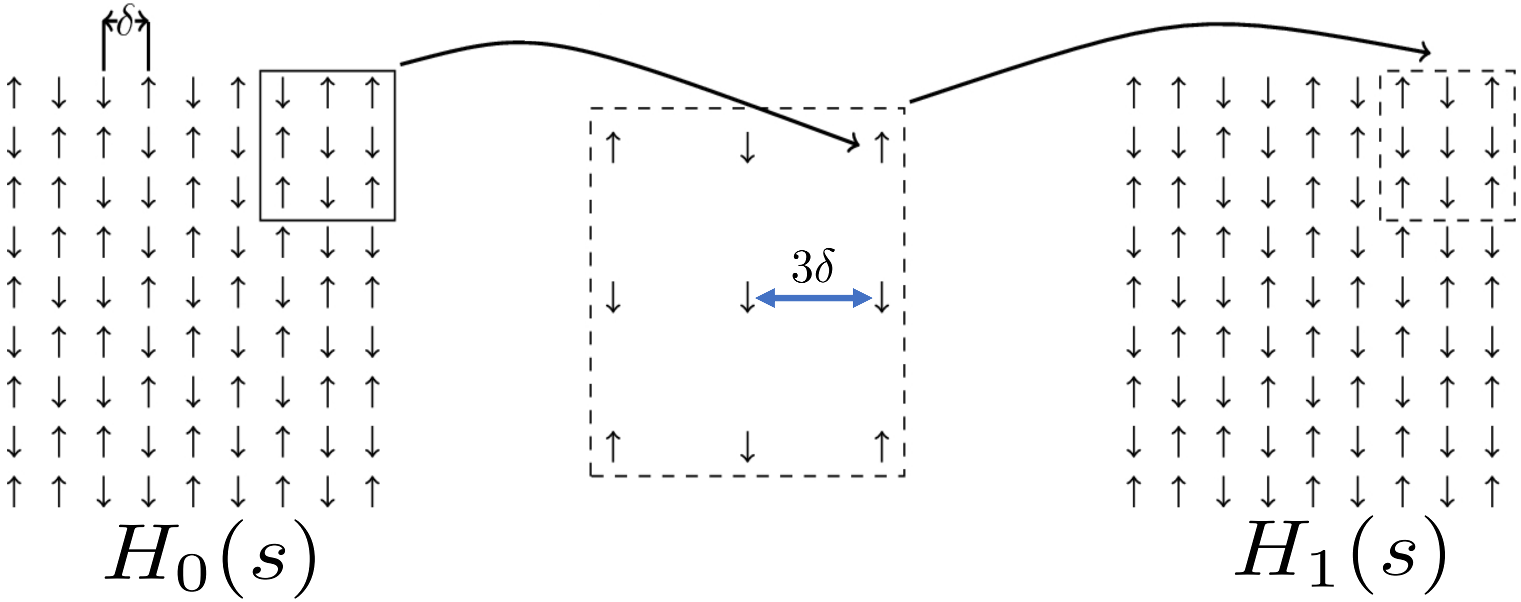
is thus expected to have fluctuations of order $\frac{1}{\sqrt{N_{BS}}} = L^{-\beta/\nu}$

Block-spins are not independent, or we would recover the standard CLT

Can we compute the corresponding stable law?

Renormalization group

Fundamental understanding of critical phenomena: renormalization group (block spins)



At criticality, $H_n \to H_*$ fixed point Hamiltonian for block-spin variables

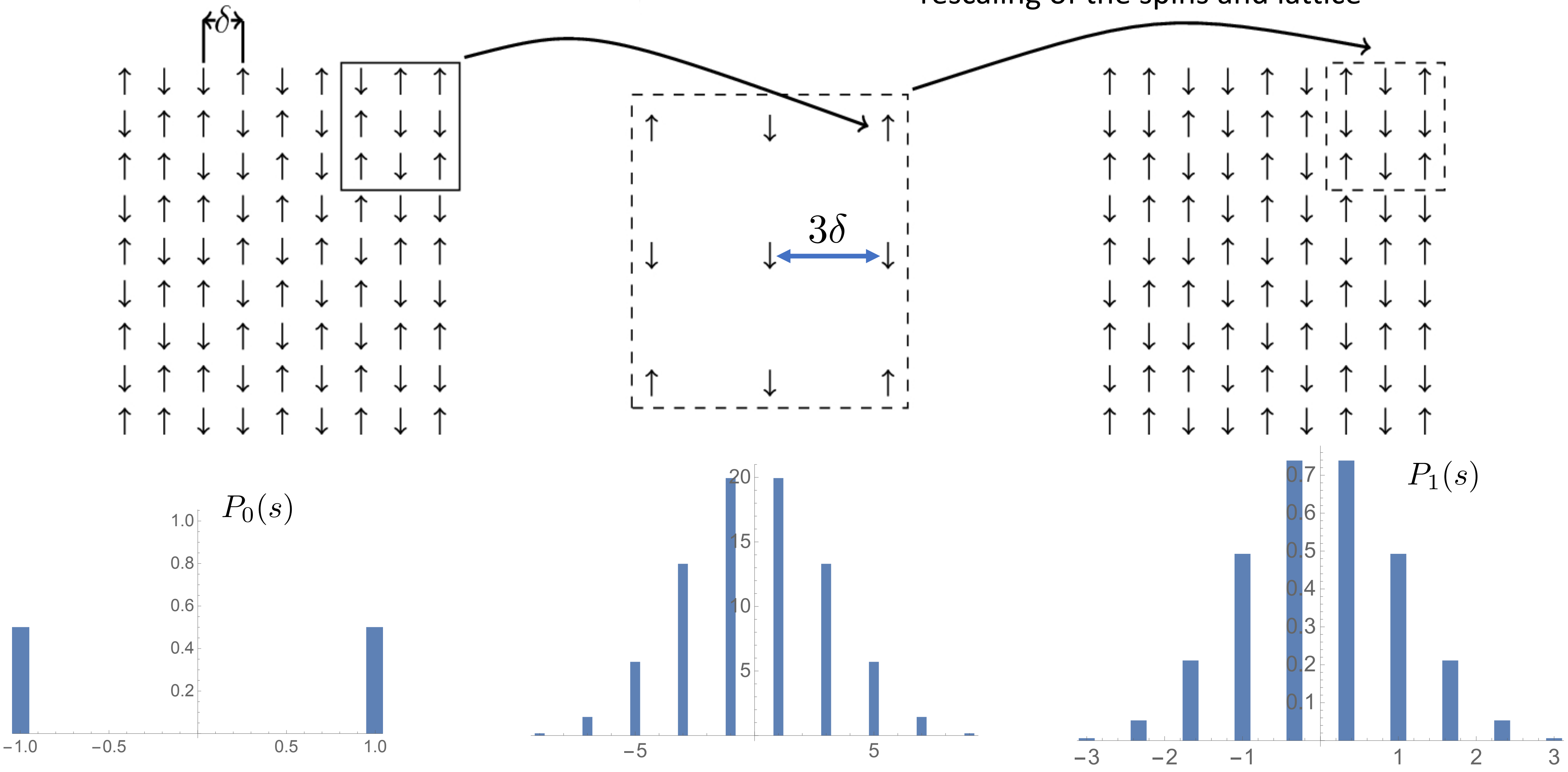
Jona-Lasino '75: RG transformation induces a flow of the probability distribution $P_n \sim e^{-H_n}$

Non critical theory = standard Central Limit Theorem

(non-trivial) critical point = generalized CLT

Example: CLT from blocking

For independent variables

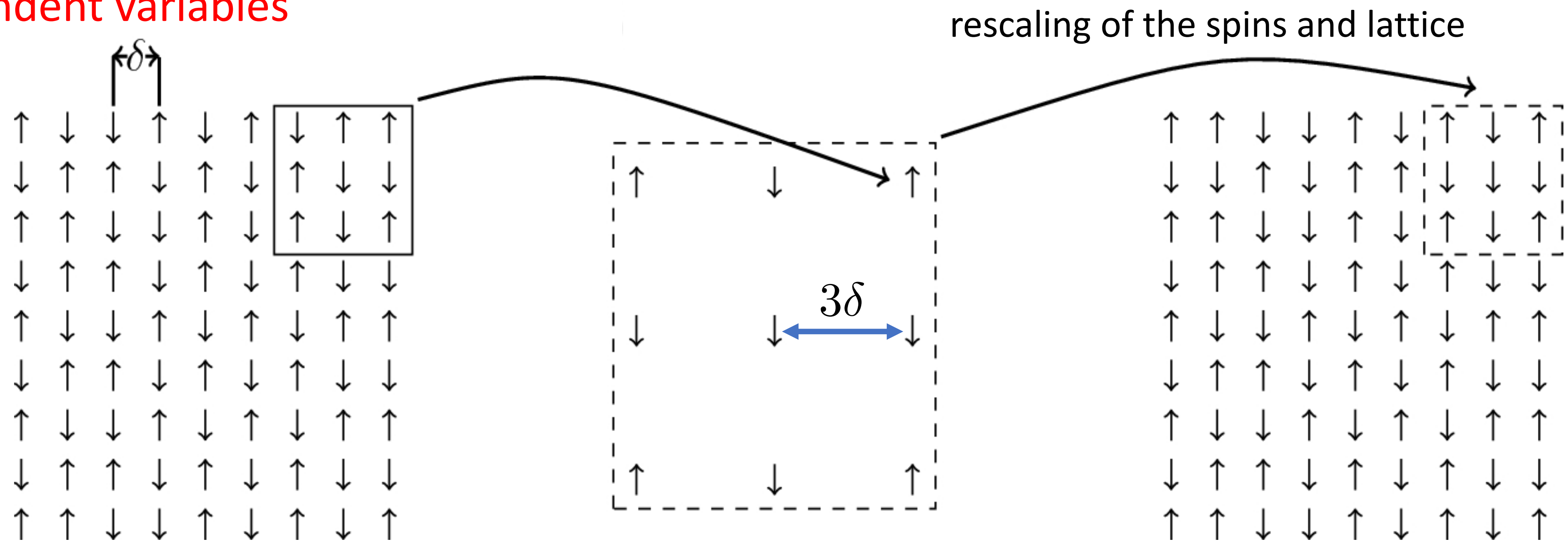


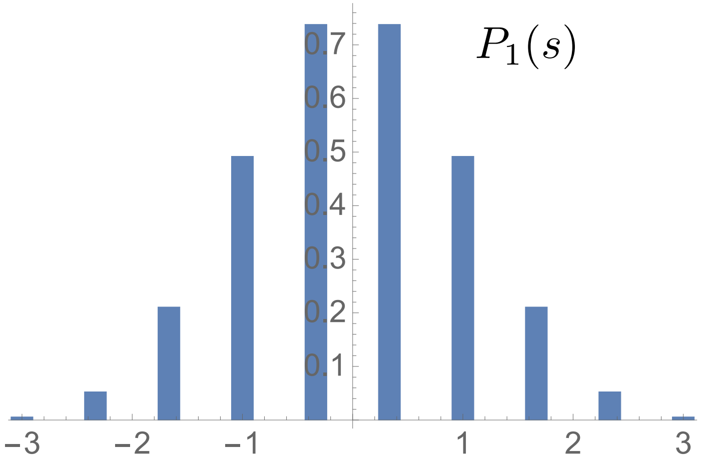
$$S_I = \sum_{i \in I} s_i$$

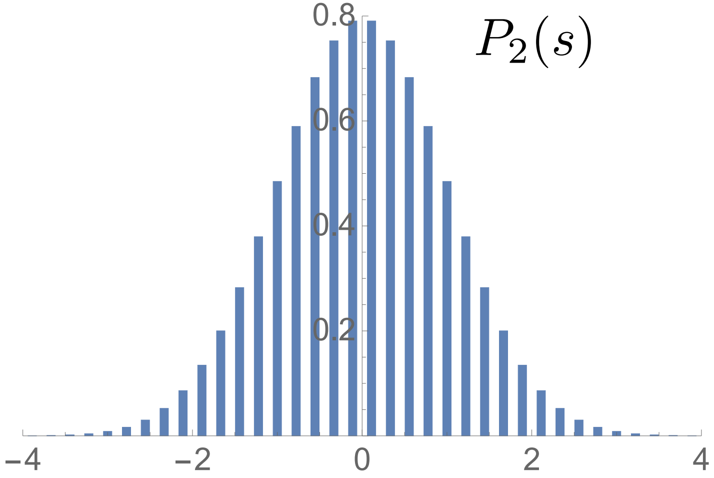
$$s_I = S_I/\sqrt{9}$$

Example: CLT from blocking

For independent variables

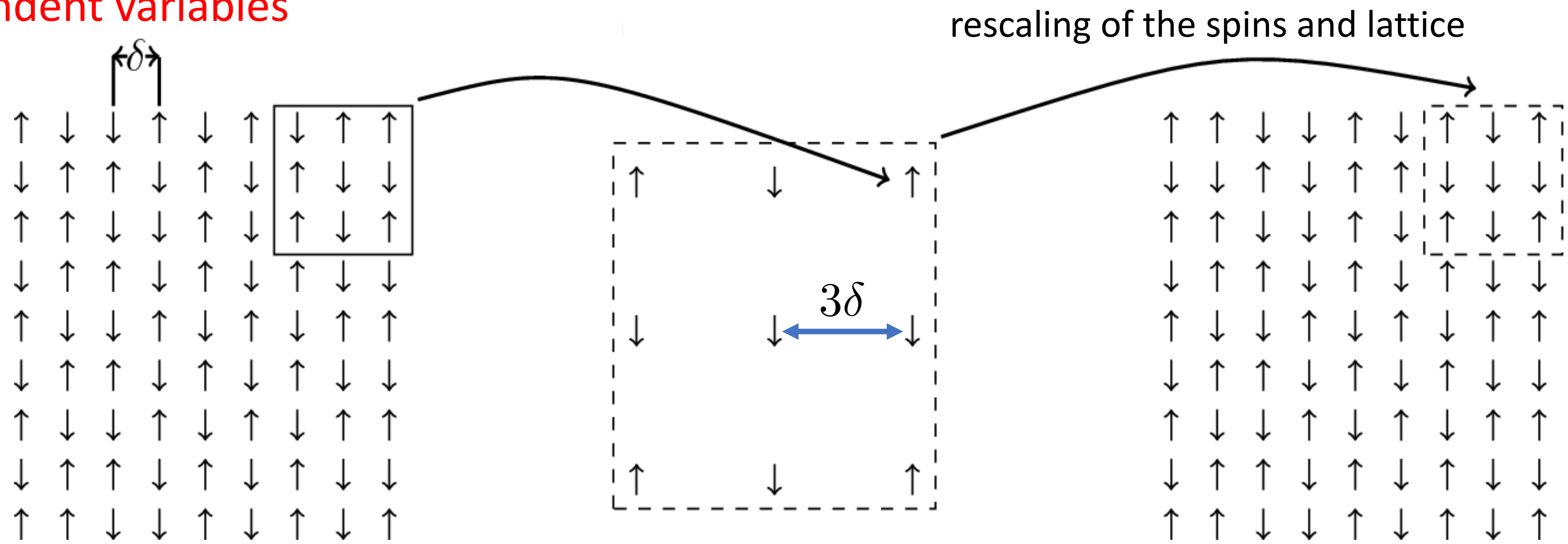




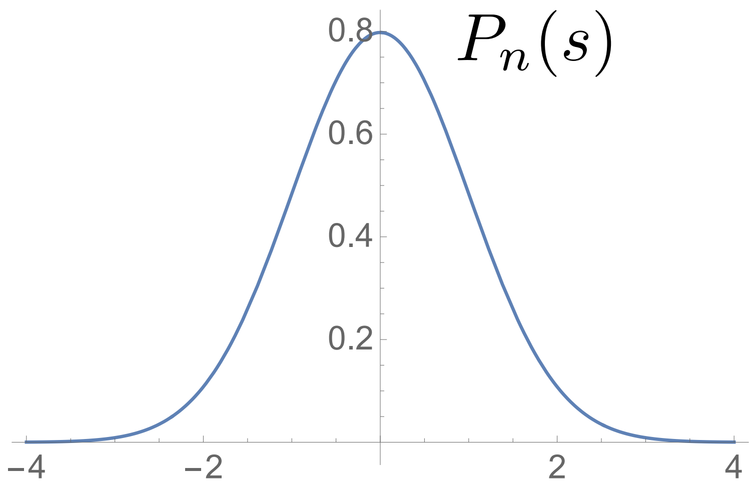


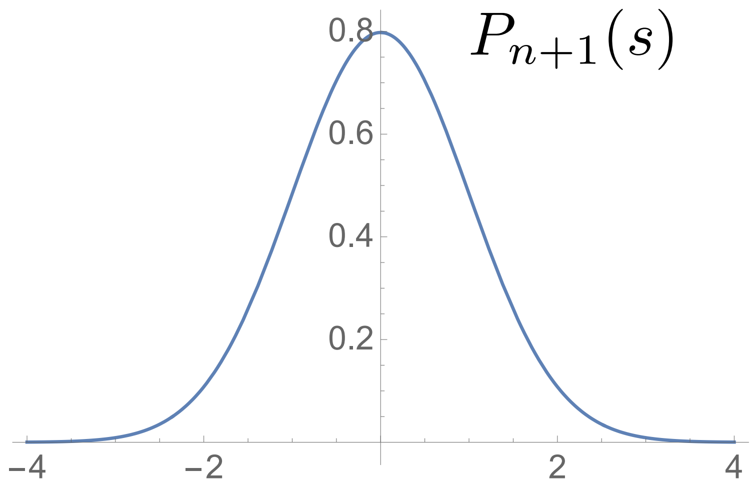
Example: CLT from blocking

For independent variables



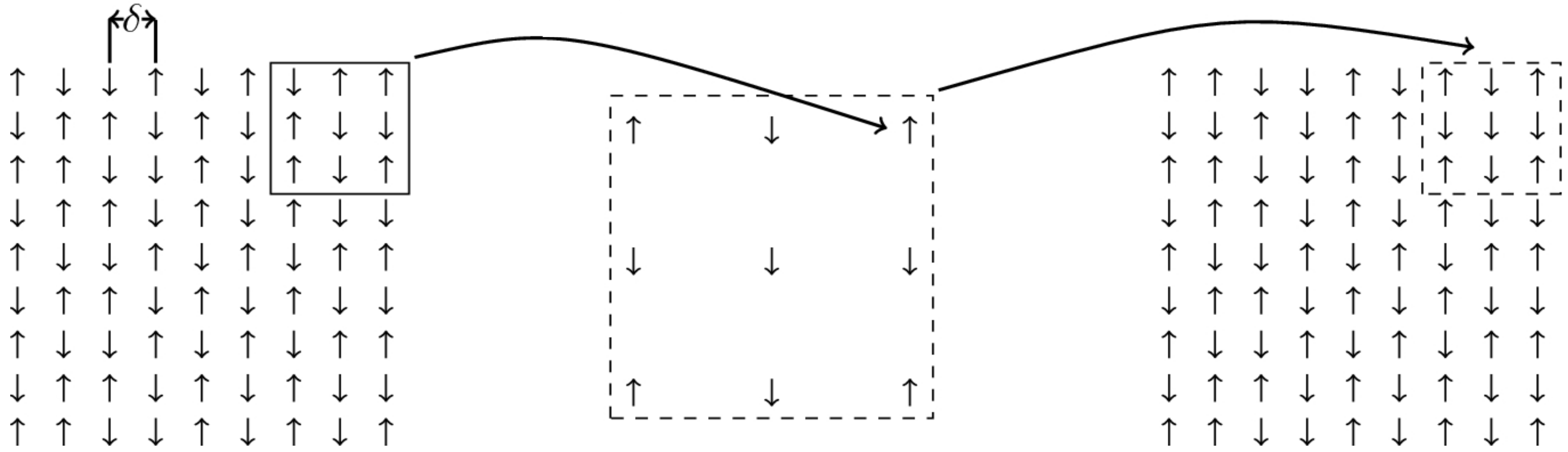
After many iterations: fixed point solution





Generalized CLT from blocking?

At criticality: $H_n \to H_*$

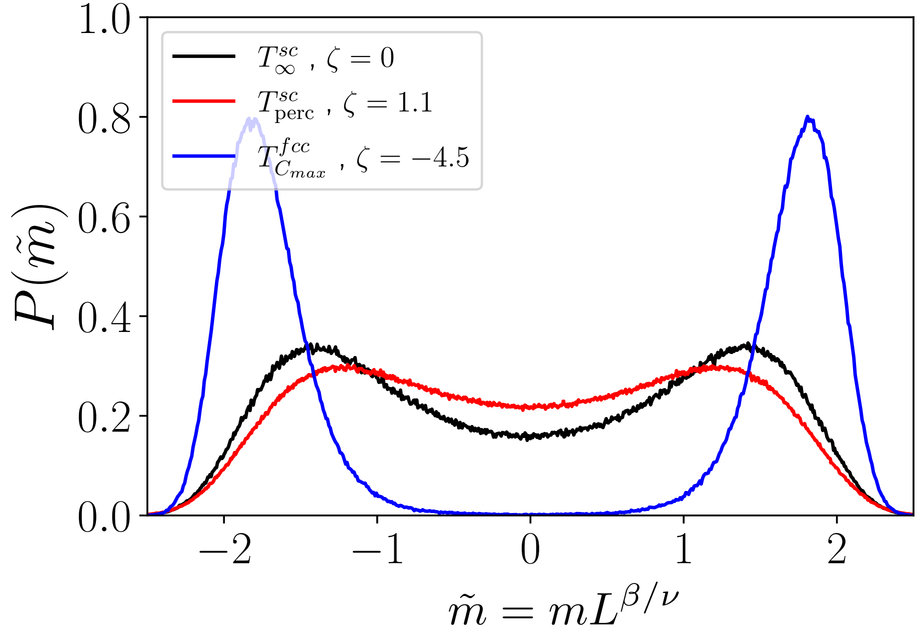


Expectation $P_* \propto e^{-H_*}$ Bruce '81, Domb and Chen '96, Rudnik et al. '98,…

Problems: - H_* depends on blocking procedure

- family of probability depending on $\zeta = L/\xi$

as $L, \xi \to \infty$

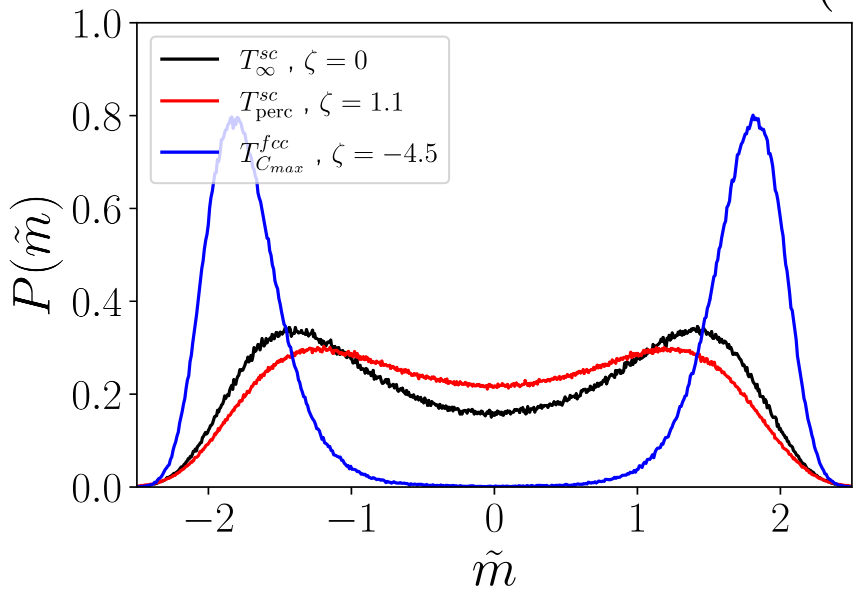


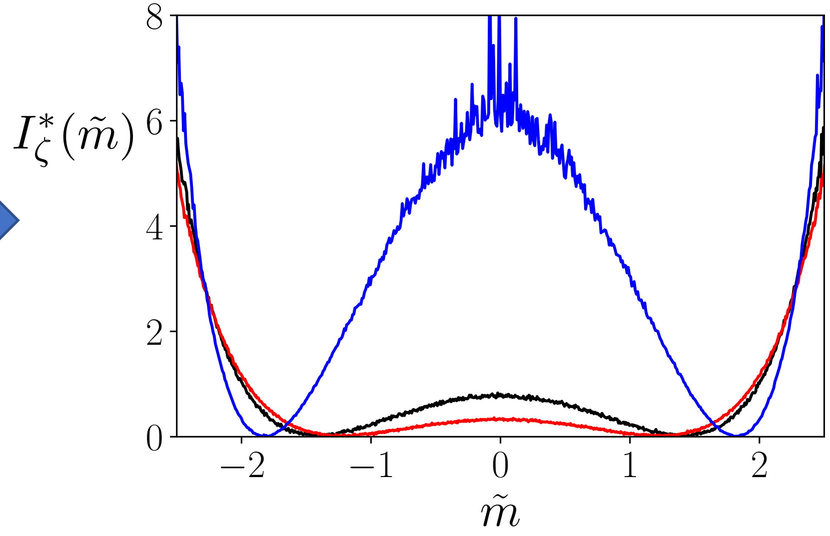
Universality of critical PDF: rate function

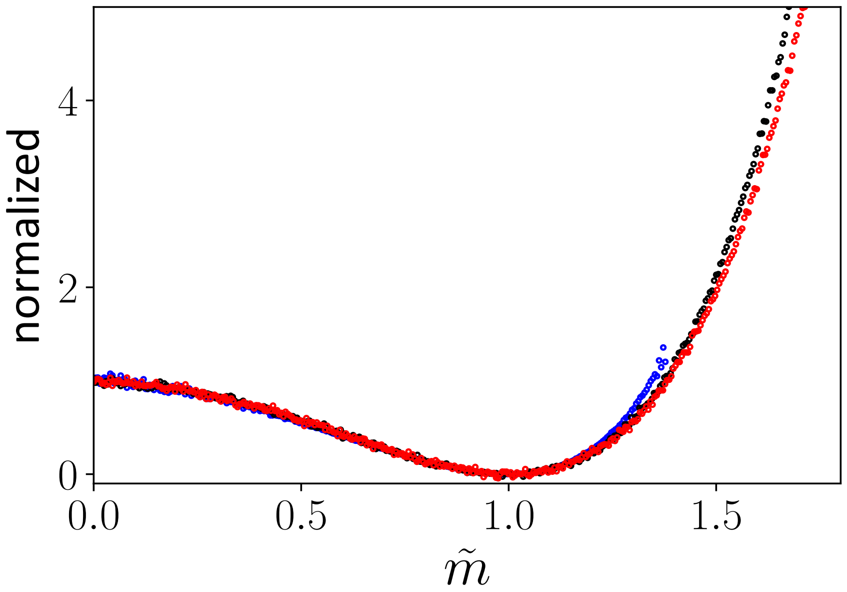
Choose $T(L) \to T_c$, such that $\zeta = L/\xi(T(L))$ is constant.

$$\tilde{m} = mL^{\beta/\nu}$$

$$P(m) \propto e^{-L^d I(m,\xi,L)} = e^{-I^*_\zeta(\tilde{m})}$$







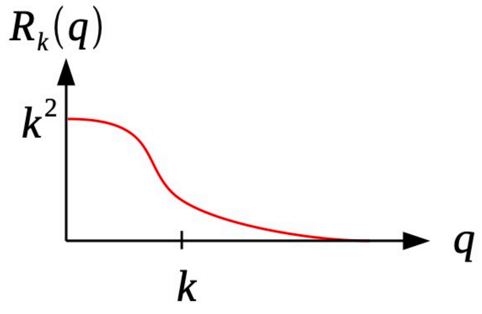
Family of rate functions
Similar to a free energy

Functional RG

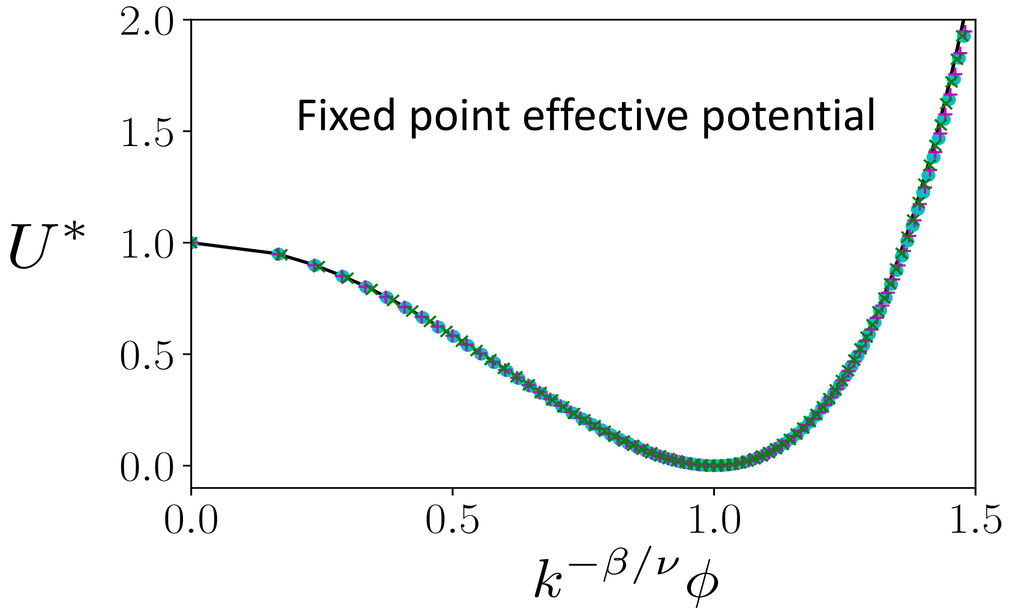
Scale-dependent Gibbs energy

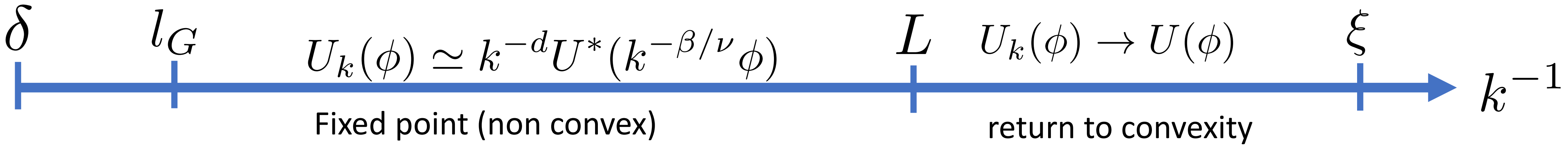
(Wetterich '93)

$$e^{-\Gamma_k[\phi]} = \int \mathcal{D}\varphi e^{-H[\varphi]-\frac{1}{2}(\phi-\varphi).R_k.(\phi-\varphi)+\frac{\delta\Gamma_k}{\delta\phi}.(\varphi-\phi)}$$



$$\partial_k\Gamma_k = \frac{1}{2}\mathrm{Tr}\left(\partial_k R_k(\Gamma_k^{(2)}+R_k)^{-1}\right)$$





Gives very good estimates of critical exponents, phase diagrams, generalizable to many other problems (out-of-eq., quantum systems, high-energy physics, etc.)

Functional for the rate function

Define a "flowing" probability distribution and rate function

$$P_k(m) = \exp(-L^d I_k(m)) = \int D\varphi\, \delta\left(m - L^{-d}\int_x \varphi\right) \exp(-H(\varphi) - 1/2\varphi.R_k.\varphi)$$

Problem: flow of the rate function is not closed

Define $\exp(-\hat{\Gamma}_k[\phi]) = \int D\varphi \exp\left(-H(\varphi) - 1/2(\phi-\varphi).\hat{R}_k.(\phi-\varphi) + \frac{\delta\hat{\Gamma}_k}{\delta\phi}.(\varphi-\phi)\right)$

with $\hat{R}_k(q) = \begin{cases} \infty, & \text{if } q = 0, \\ R_k(q), & \text{if } q > 0. \end{cases}$

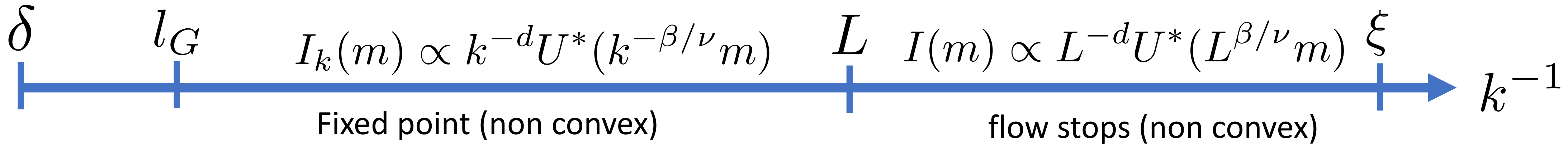
Flow equation for $\hat{\Gamma}_k[\phi]$ closed and $I_k(m) = \hat{\Gamma}_k[\phi = m]$

Flow equation for the rate function

Simple approximation: “Local Potential Approximation”

$$\partial_k I_k(m) = \frac{1}{2L^d} \sum_q \frac{\partial_k \hat{R}_k(q)}{q^2 + \hat{R}_k(q) + I_k''(m)}$$

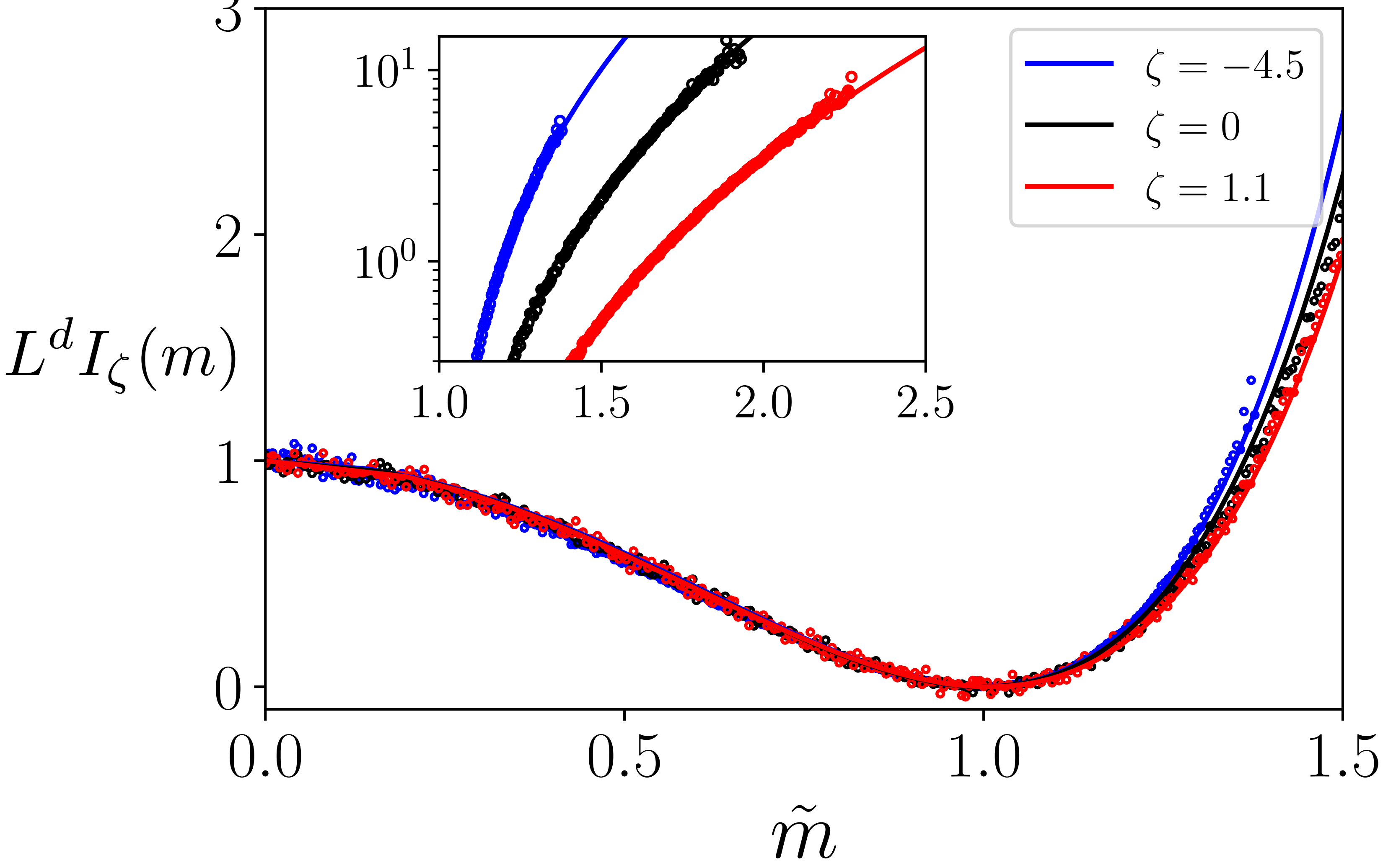
Non-linear partial differential equation, boundary layers, cusps, shocks



Results

Rate function from modified FRG (Local Potential Approximation)

No fitting parameters (beyond normalization)



Conclusions

- Renormalization group is a natural framework to understand strongly correlated variables

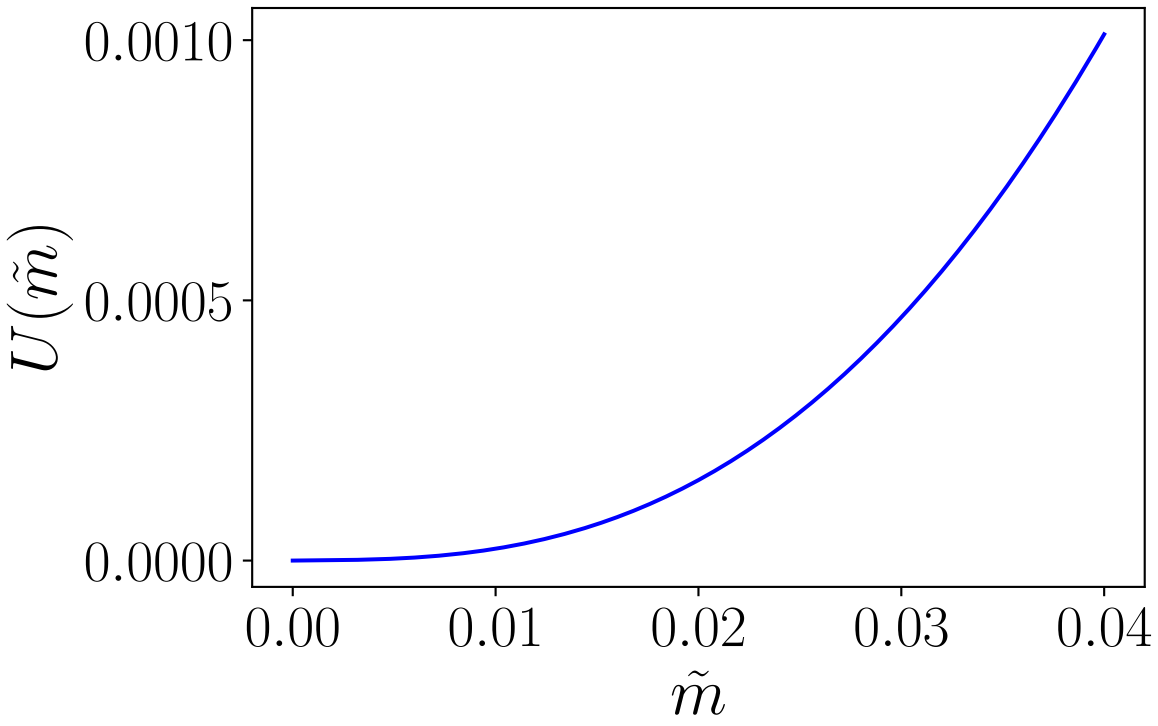
- Modified FRG is an explicit construction from first principle of the rate function

- Rate function controlled by the fixed point potential of FRG

- Beyond the Ising model: critical dynamics, KPZ, quantum systems…

- Allows for explicit computation of "mesoscopic/low energy" effective models, based on systematically improvable and convergent approximations

Return to convexity



Probability distribution and RG fixed points

Direct connection between the probability distribution and RG fixed point quantities we compute?

$$H(\varphi) = \int d^dx \, ((\nabla\varphi)^2 + r\varphi^2 + g\varphi^4)$$

Wilson's RG

$$H^*(\varphi) = \int d^dx \, ((\nabla\varphi)^2 + r^*\varphi^2 + g^*\varphi^4 + \cdots)$$

Fixed point coupling constants

Link between Wilson's RG and probability distribution:
Bruce '81, Domb and Chen '96, Rudnik et al. '98,…

$$P(m) \propto e^{-H^*(m)}$$???

Based on perturbative RG, inclusion of finite size ad hoc…
and fixed point couplings are scheme dependent!?!

Template