CEMPI DAYS

”From wave turbulence to integrable turbulence and soliton gases”

Stéphane Randoux (PhLAM)

Lille, April 1st (2022)

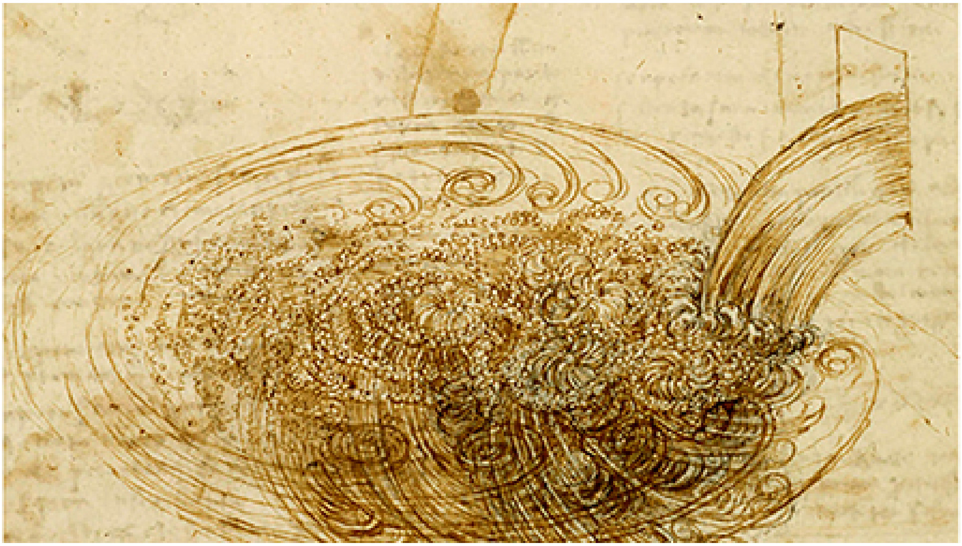
anr

Université de Lille

CEMPI CENTRE EUROPÉEN POUR LES MATHÉMATIQUES, LA PHYSIQUE ET LEURS INTERACTIONS

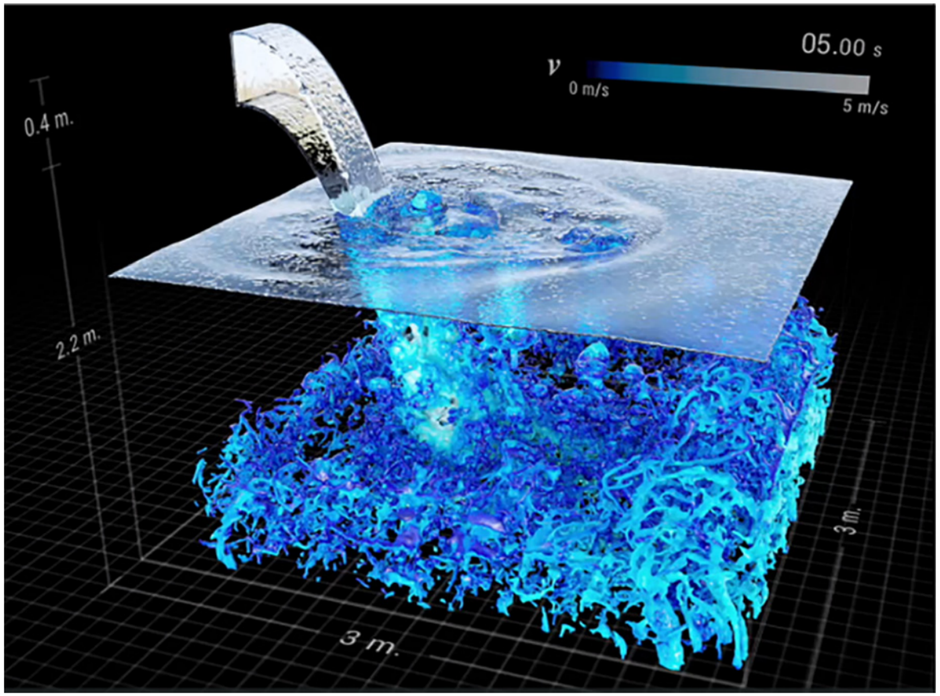
PhLAM Physique des Lasers Atomes et Molécules

cnrs

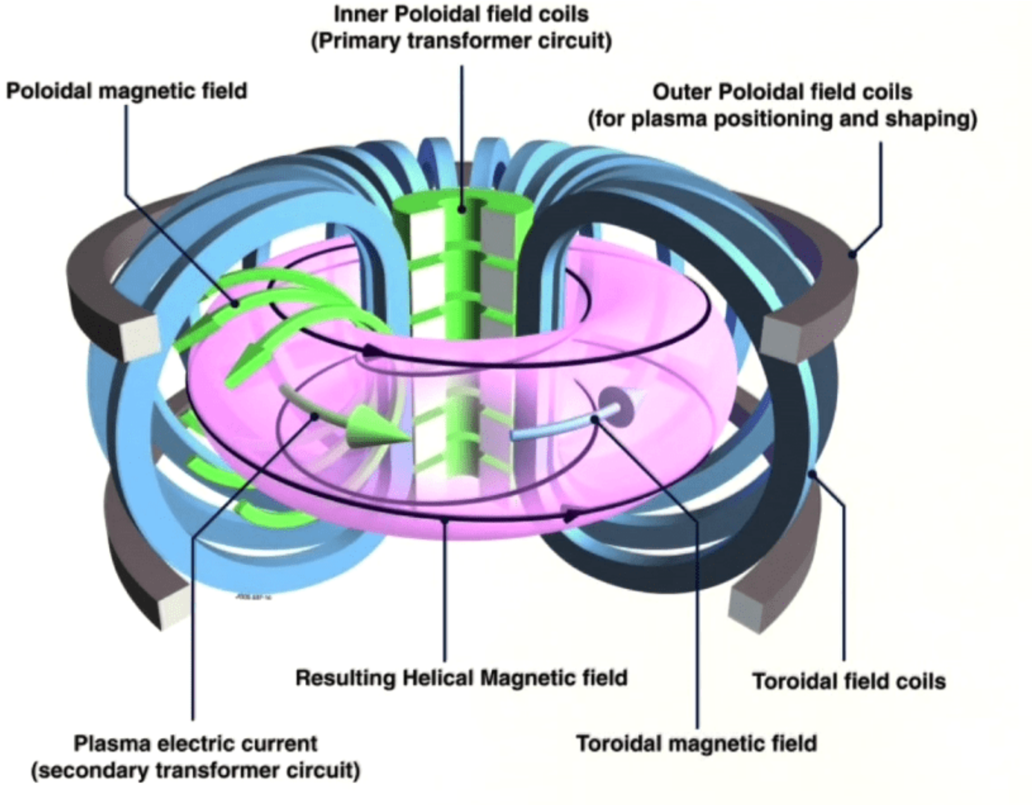


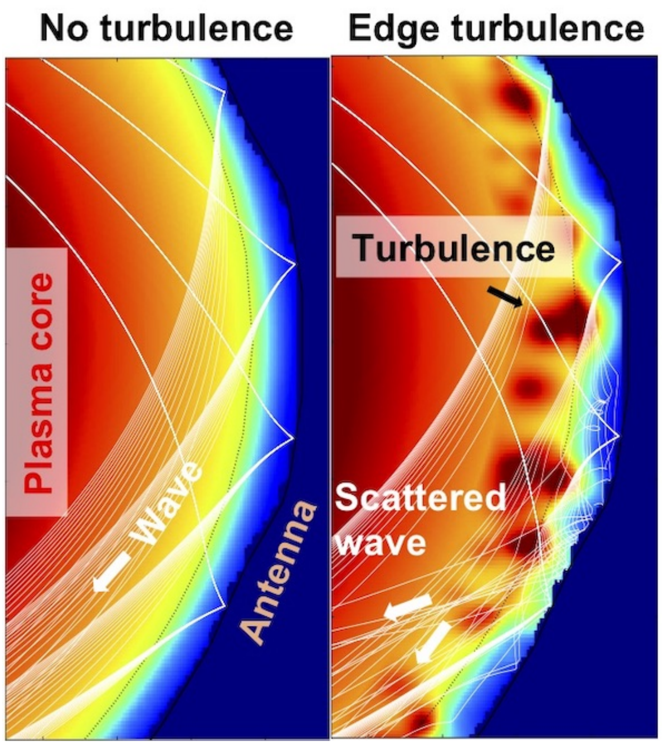
Leonardo da Vinci (~ 1510)
The fall of a stream of
water from a sluice into a pool.

"Da Vinci's observation of turbulence: A French-Italian study aiming at numerically reproducing the physics behind one of his drawings, 500 years later"; Colagrossi et al, Phys. Fluids **33**, 115122 (2021)



(Wave) turbulence in Tokamak

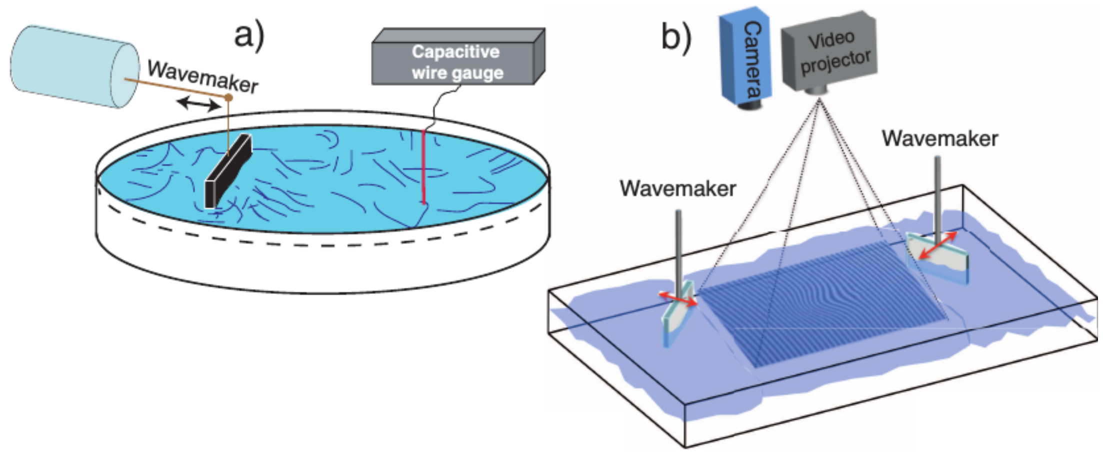




PRL **119,** 221101 (2017) PHYSICAL REVIEW LETTERS week ending 1 DECEMBER 2017

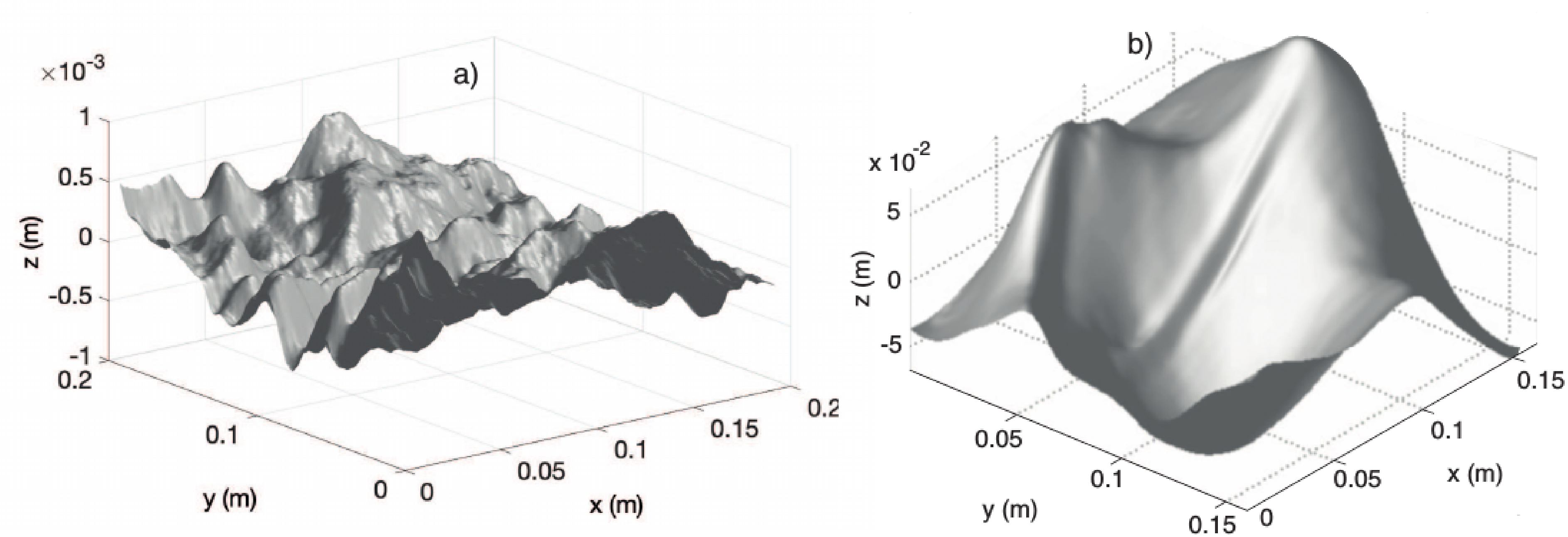
Turbulence of Weak Gravitational Waves in the Early Universe

Sébastien Galtier[1,*] and Sergey V. Nazarenko[2,†]
[1]*Laboratoire de Physique des Plasmas, École Polytechnique, Univ. Paris-Sud, Université Paris-Saclay, F-91128 Palaiseau Cedex, France*
[2]*Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom*



Small forcing

Strong forcing



Experiments in surface gravity-capillary waves, E. Falcon and N. Mordant, Ann. Rev. Fluid. Mech. (2022)

$$i\frac{\partial a_{\boldsymbol{k}}}{\partial t} = \frac{\partial H}{\partial a^*_{\boldsymbol{k}}}$$

$a_{\boldsymbol{k}}$ Fourier mode of the wave field

$\omega = \omega(\boldsymbol{k})$ Dispersion relation

$$i\frac{\partial a_{\mathbf{k}}}{\partial t} = \frac{\partial H}{\partial a^*_{\mathbf{k}}} = \omega a_{\mathbf{k}} + \varepsilon \int V_{k,k_1,k_2} a_{\mathbf{k}_1} a_{\mathbf{k}_2} \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) d\mathbf{k}_1 \mathbf{k}_2$$
$$+\varepsilon^2 \int W_{k,k_1,k_2,k_3} a_{\mathbf{k}_1} a_{\mathbf{k}_2} a_{\mathbf{k}_3} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}) d\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3 + \ldots,$$

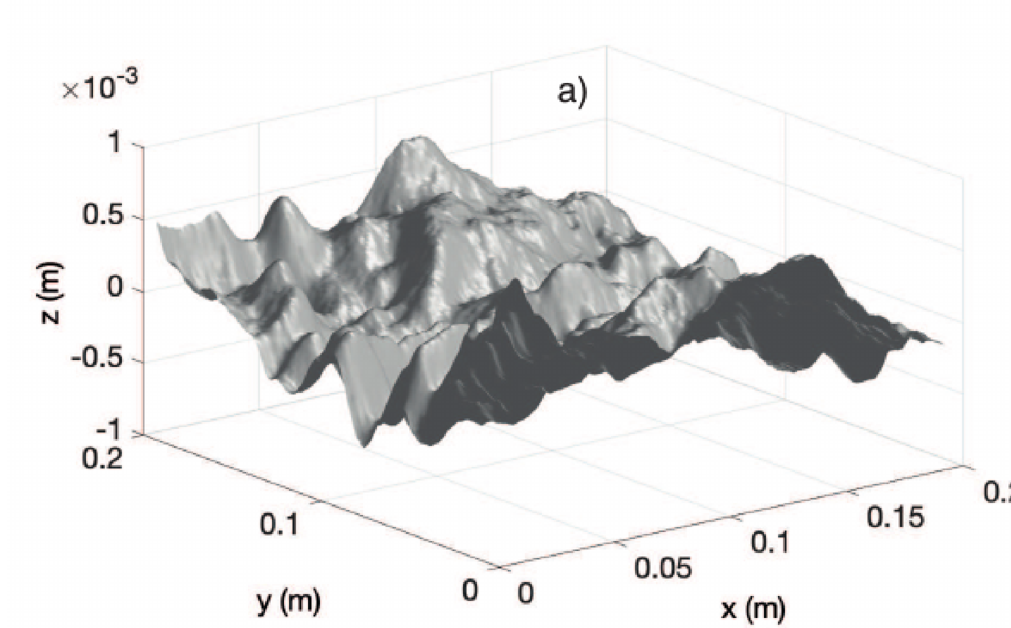
$$n_{\boldsymbol{k}} = < a_{\boldsymbol{k}} a^*_{\boldsymbol{k}} >$$

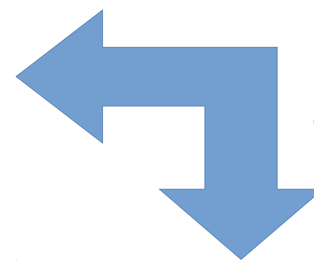
Kinetic equation (4-wave interaction)

$$\frac{\partial n_{\mathbf{k}}}{\partial t} = 4\pi\varepsilon^4 \int |W_{k,k_1,k_2,k_3}|^2 n_{\mathbf{k}} n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \left[\frac{1}{n_{\mathbf{k}}} + \frac{1}{n_{\mathbf{k}_1}} - \frac{1}{n_{\mathbf{k}_2}} - \frac{1}{n_{\mathbf{k}_3}} \right]$$
$$\delta(\omega + \omega_1 - \omega_2 - \omega_3) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3$$

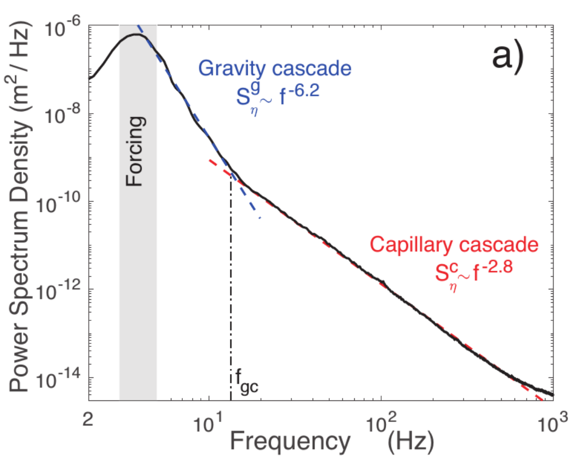
S. Nazarenko, “Wave Turbulence”, Springer Science (2011)

Wave Turbulence



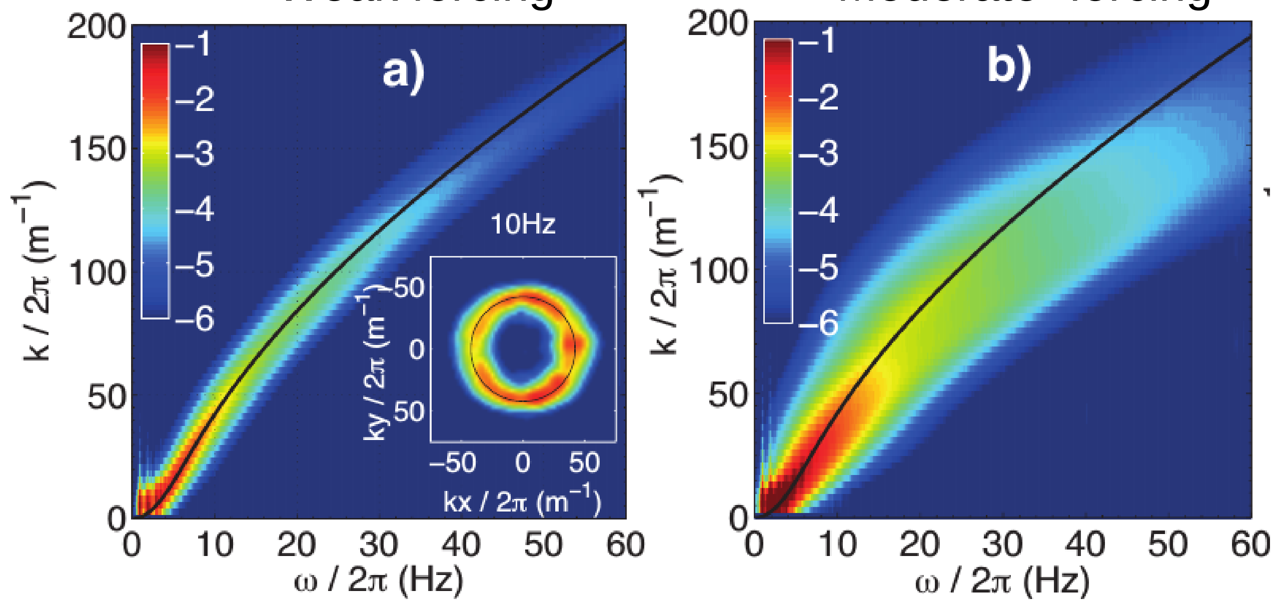


2D Fourier Transform

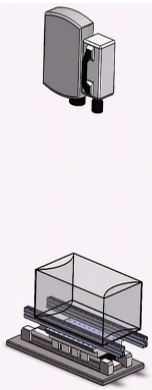


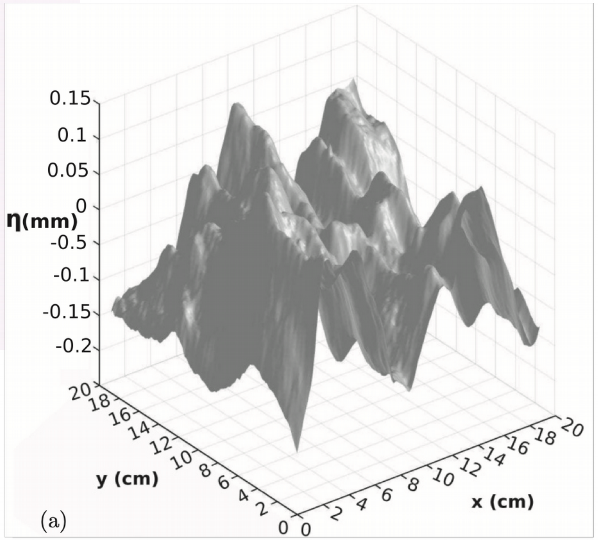
Weak forcing

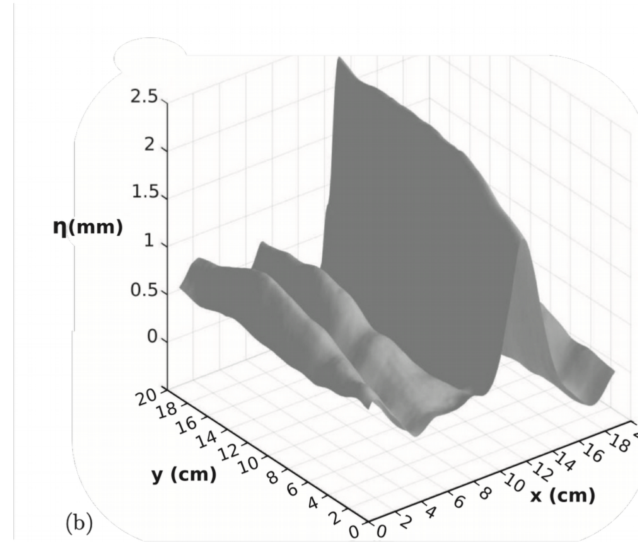
“Moderate” forcing



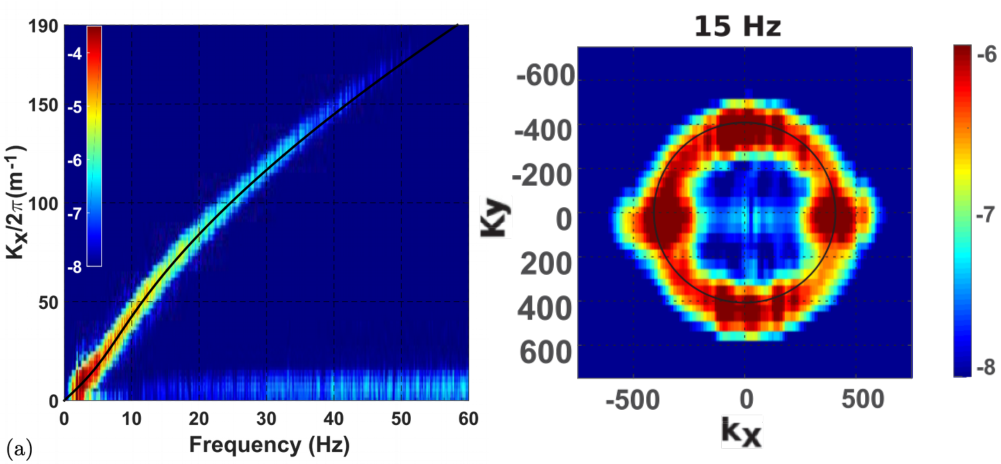
Experiments in surface gravity-capillary waves, E. Falcon and N. Mordant, Ann. Rev. Fluid. Mech. (2022)



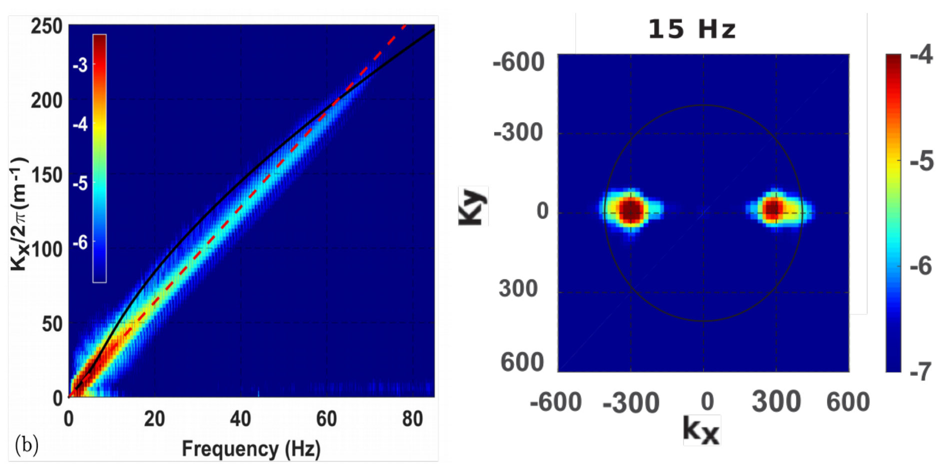




weak forcing amplitude: 0.9 mm

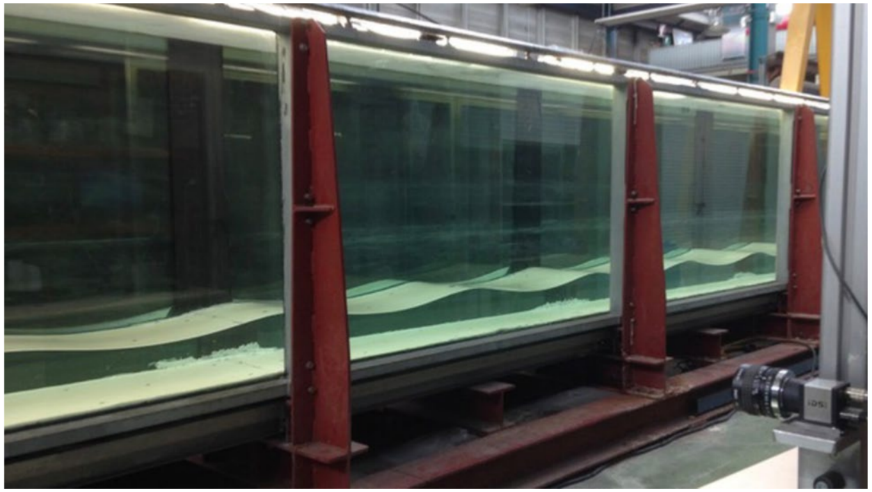


stronger forcing amplitude: 2.5 mm

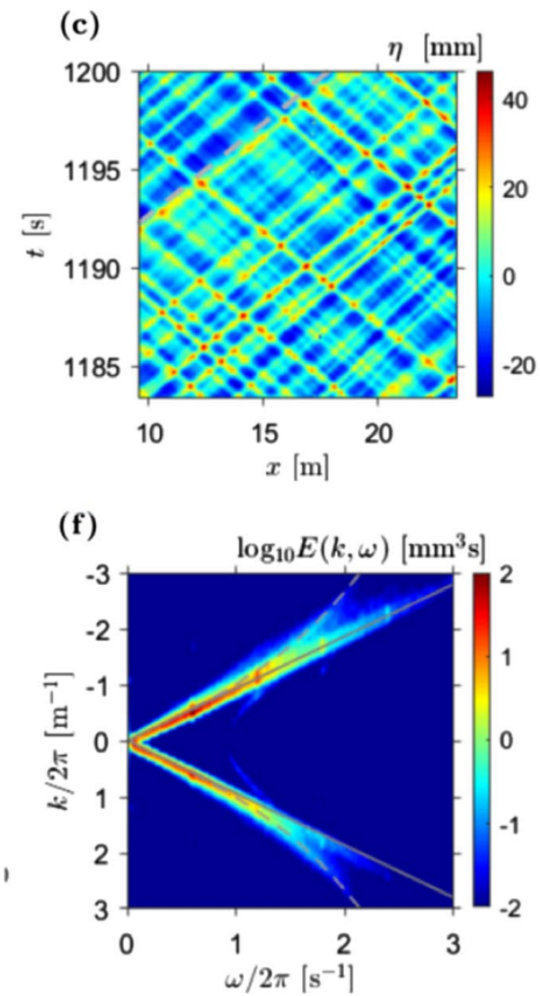


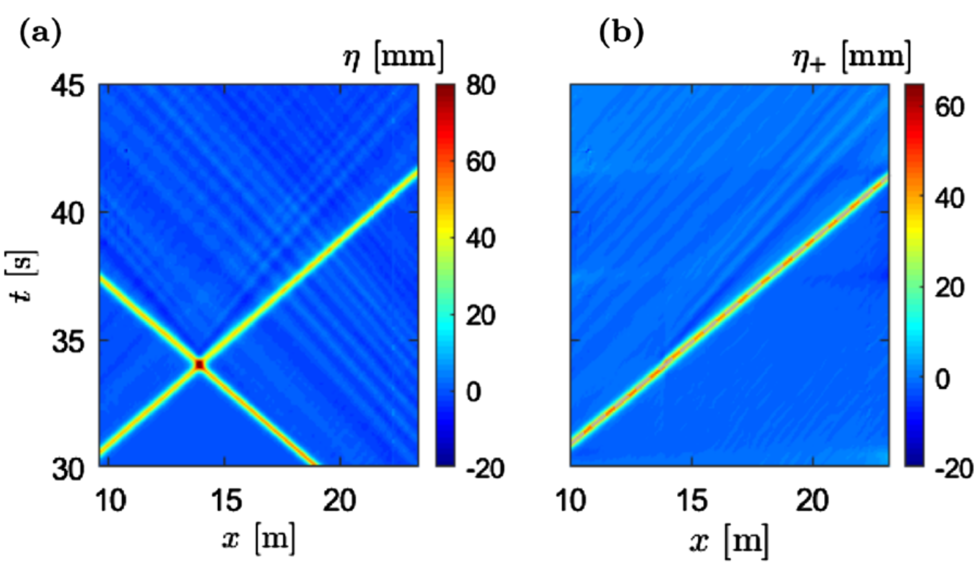
“Transition from weak wave turbulence to soliton gas” R. Hassaini and N. Mordant Phys. Rev. Fluids 2, 094803 (2017)

Soliton gas in shallow water



1D experiments
33-m long tank in Grenoble
Shallow water experiments (KdV dynamics)

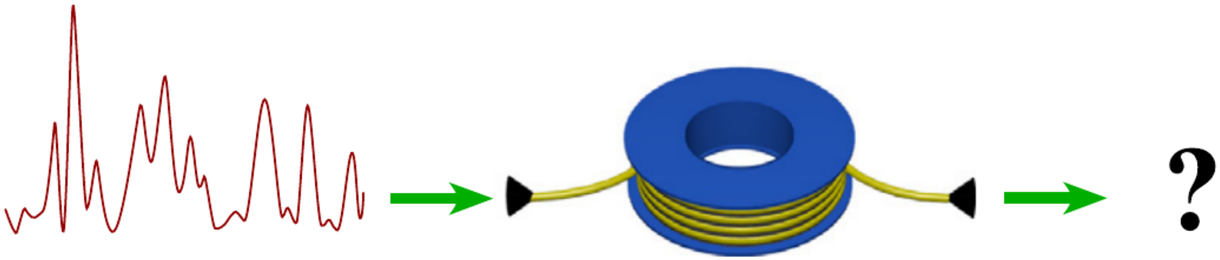




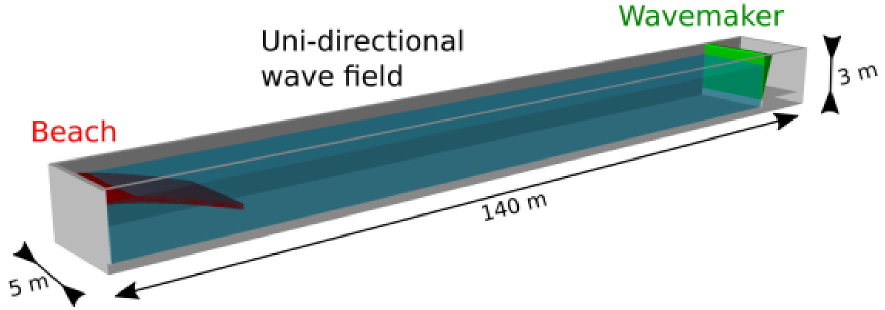
Experimental evidence of a hydrodynamic soliton gas, Y. Redor et al, Phys. Rev. Lett. 122, 214502 (2019)

« Nonlinear wave systems integrable by **Inverse Scattering Transform (IST**) method could demonstrate a complex behavior that demands the statistical description. The theory of this description composes a new chapter in the theory of wave turbulence- Turbulence in Integrable Systems »

"Turbulence in Integrable Systems," V. E. Zakharov, Studies in Applied Mathematics, **122**, 219 (2009).



Experiments in Integrable Turbulence



$$u_t + 6uu_x + u_{xxx} = 0$$ KdV equation

$$iq_t + q_{xx} + 2|q|^2 q = 0$$ NLS equation

Fourier Transform for solving linear PDEs

$$iq_t = q_{xx} \qquad q(x,t=0) = exp(-x^2)$$ Gaussian input pulse

$q(x,t=0)$ —— Fourier Transform ——→ $\tilde{q}(k,t=0) = \int_{-\infty}^{+\infty} q(x,t=0)\, e^{-ikx}\, dx$

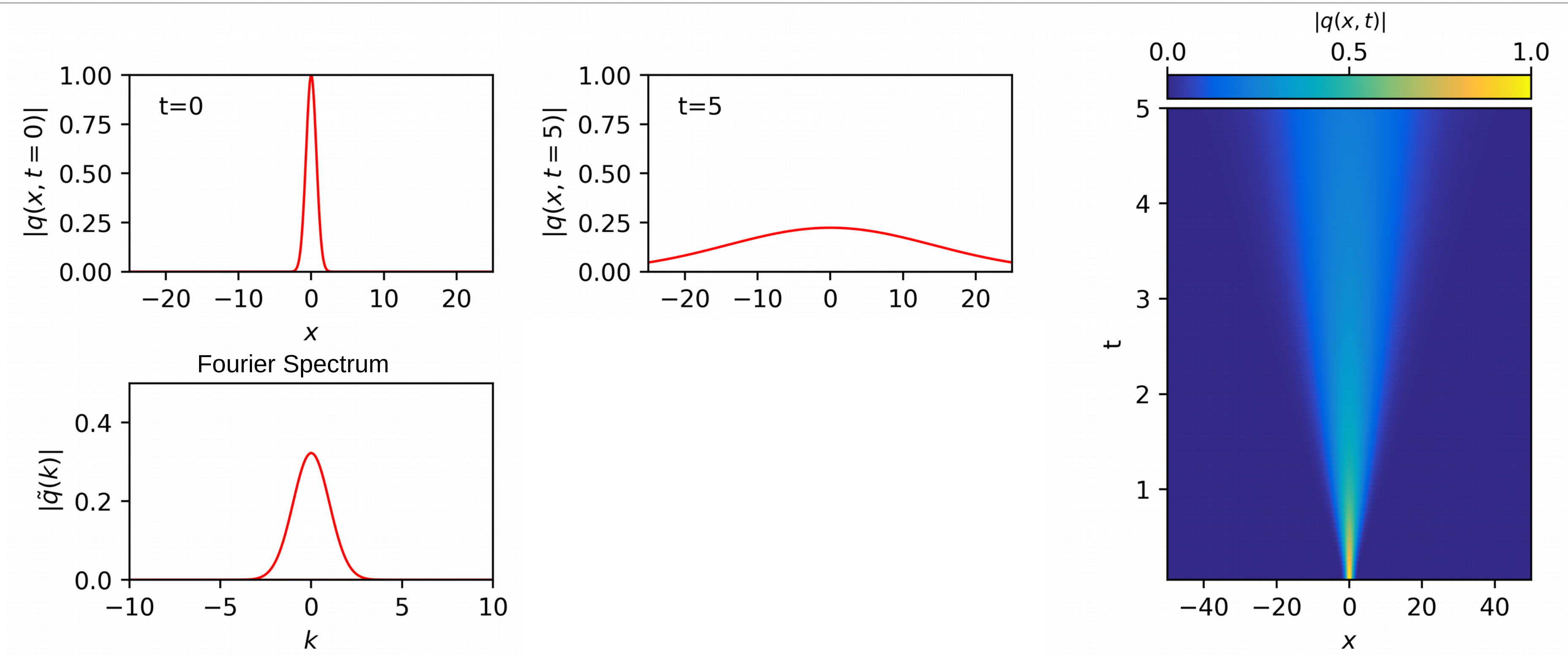
$$i\tilde{q}_t(k,t) = \omega(k)\,\tilde{q}(k,t) \qquad \omega(k) = -k^2$$

Dispersion relation

$$\tilde{q}(k,t) = \tilde{q}(k,t=0)\, e^{i\omega(k)t}$$

$q(x,t)$ ←—— Inverse Fourier Transform ——

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{q}(k,t)\, e^{ikx}\, dk$$



IST method for solving the integrable 1D-NLSE

$$iq_t + q_{xx} + 2|q|^2 q = 0$$ Focusing 1D-NLSE

$q(x, t=0)$ —— Direct scattering ——> $S(\xi, t=0)$ $\left\{ \begin{array}{l} \rho(k, t=0),\ k \in \mathbb{R} \\ [\xi_j\ ,\ C_{j,0}]_{j=1}^{N} \end{array} \right\}$

↓

$q(x,t)$ <—— Inverse scattering —— $S(\xi, t)$ $\left\{ \begin{array}{l} \rho(\xi, t) = \rho(\xi, t=0) e^{-i4\xi^2 t} \\ \left\{ \begin{array}{l} C_j(t) = C_{j,0} e^{-i\,4\,\xi_j^2\,t} \\ \xi_j = constant \end{array} \right\}_{j=1}^{j=N} \end{array} \right.$

$S(\xi)$	**Scattering data**

$\xi_j\ (j = 1 \to N)$ Discrete eigenvalues

$C_j\ (j = 1 \to N)$ Norming constants

} Discrete spectrum (solitonic content)

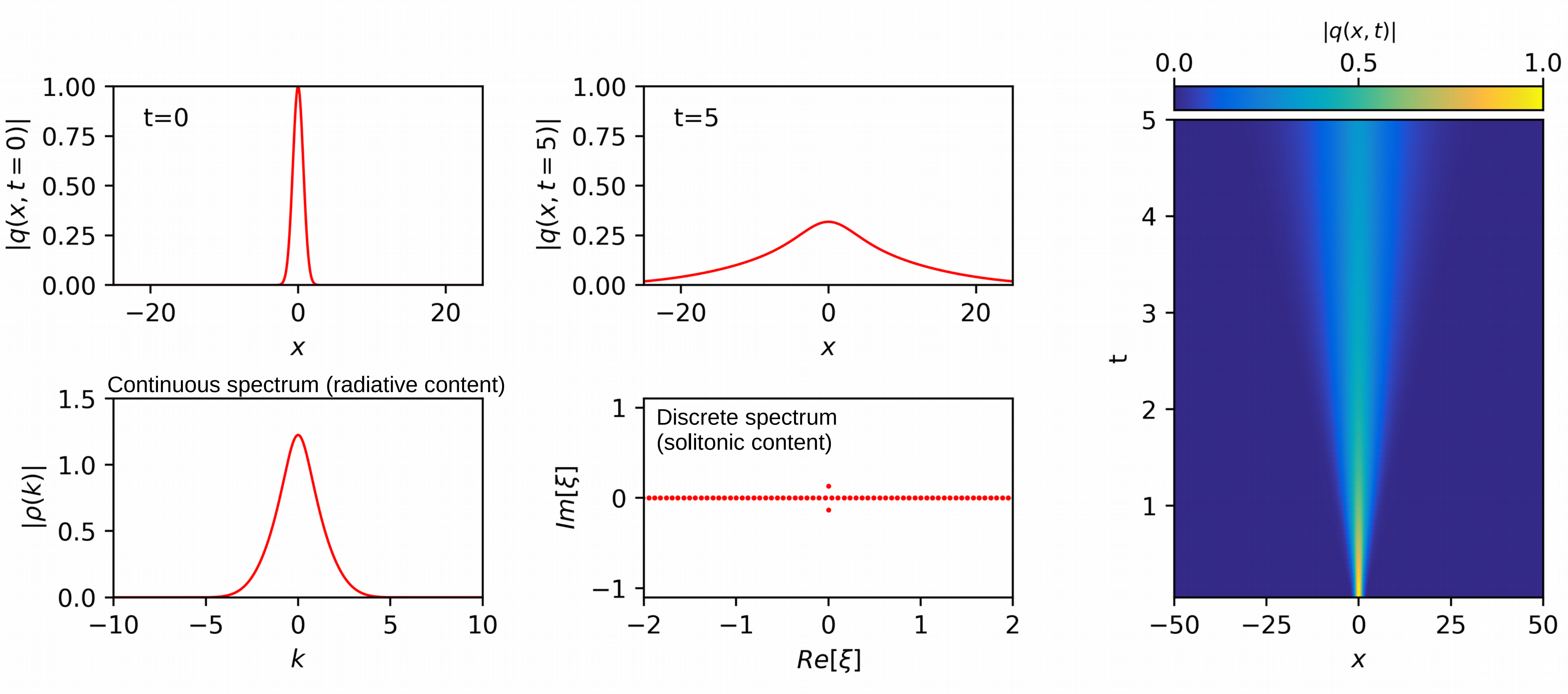
$\rho(k),\ k \in \mathbb{R}$ Continuous spectrum (radiative content)

P. G. Drazin and R. S Johnson “Solitons: an introduction” Cambridge Press in Applied mathematics (1989)

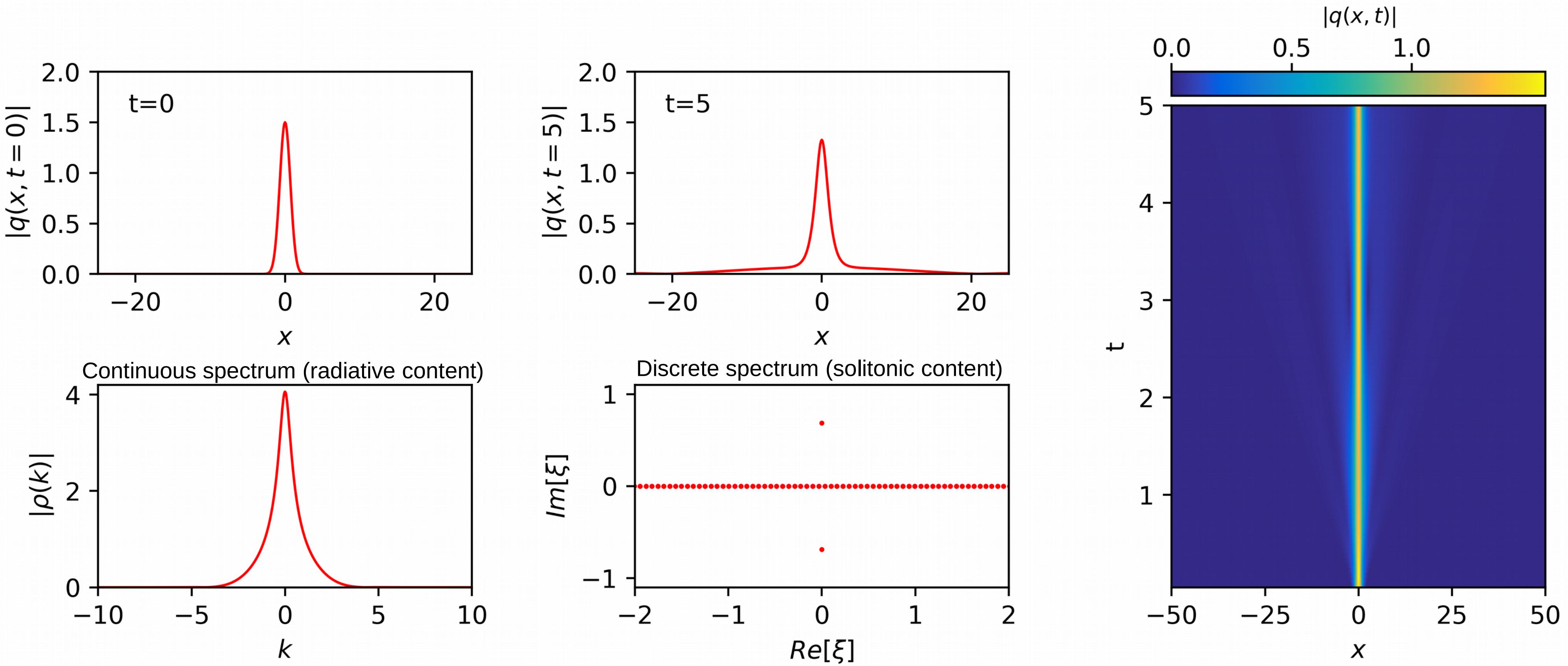
IST method for solving the integrable 1D-NLSE

$$iq_t + q_{xx} + 2|q|^2 q = 0 \qquad q(x, t=0) = exp(-x^2)$$

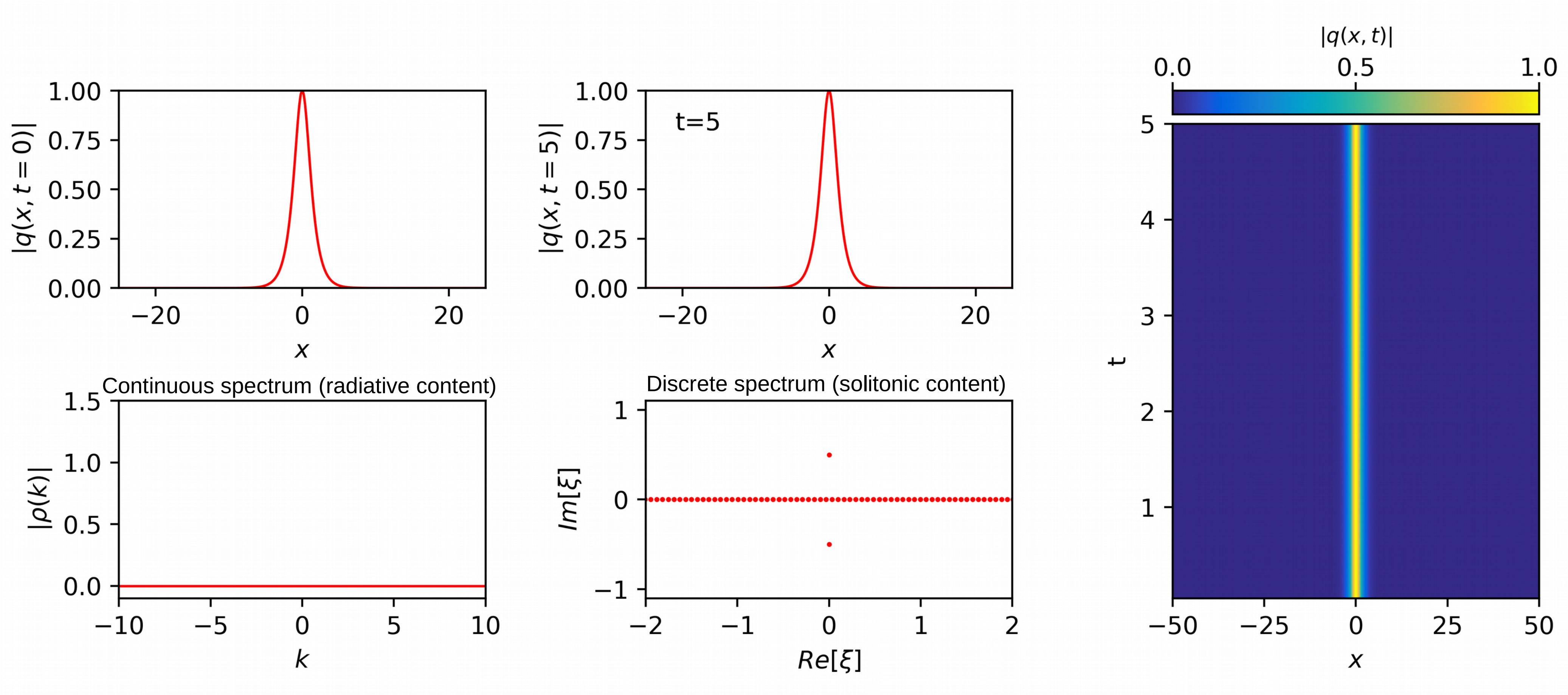
Gaussian input pulse



$$iq_t + q_{xx} + 2|q|^2 q = 0 \qquad q(x, t=0) = 1.5\, exp(-x^2)$$

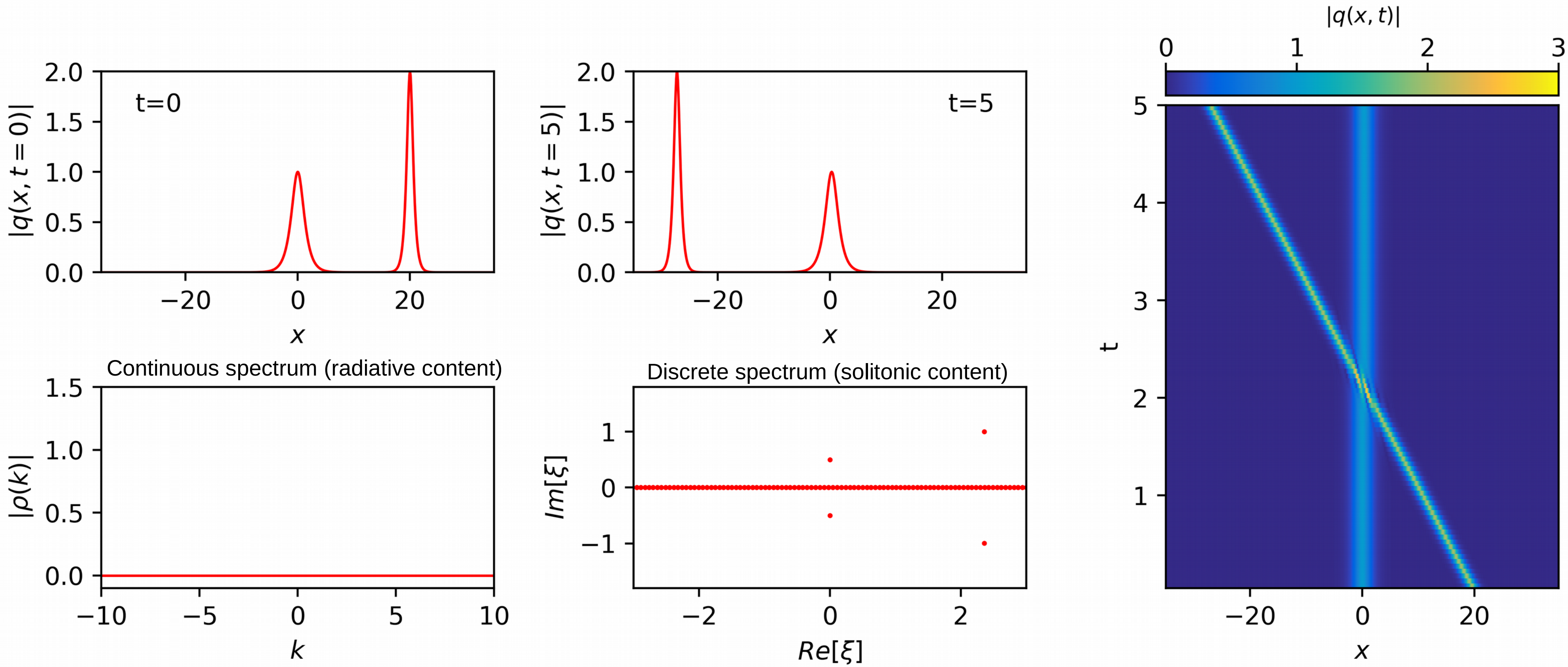


$$iq_t + q_{xx} + 2|q|^2 q = 0 \qquad q(x, t=0) = sech(x)$$



$$\xi_1 = 0.5\,i$$

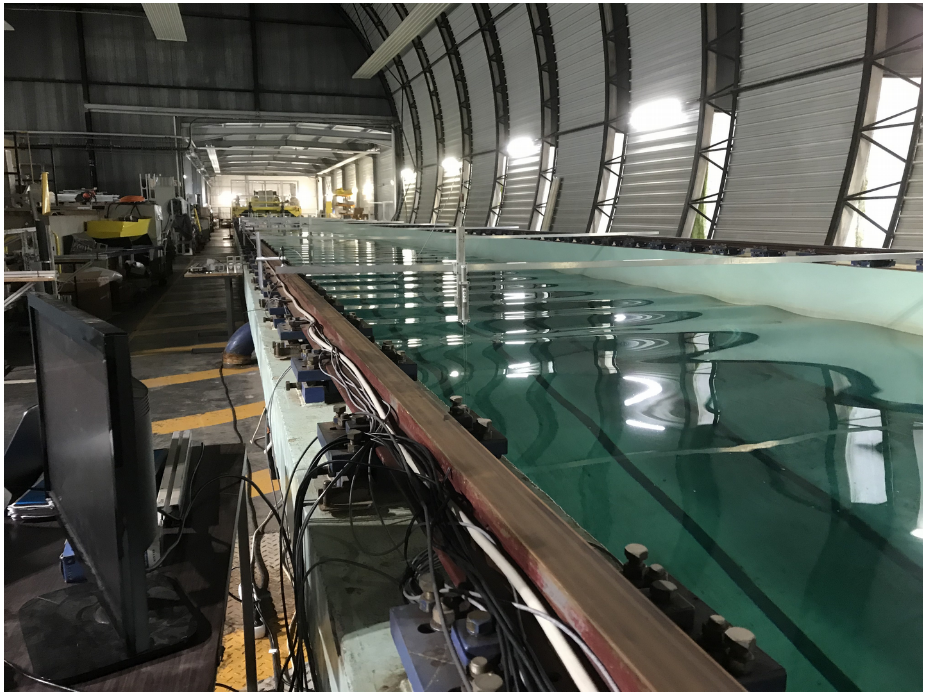
$$iq_t + q_{xx} + 2|q|^2 q = 0$$

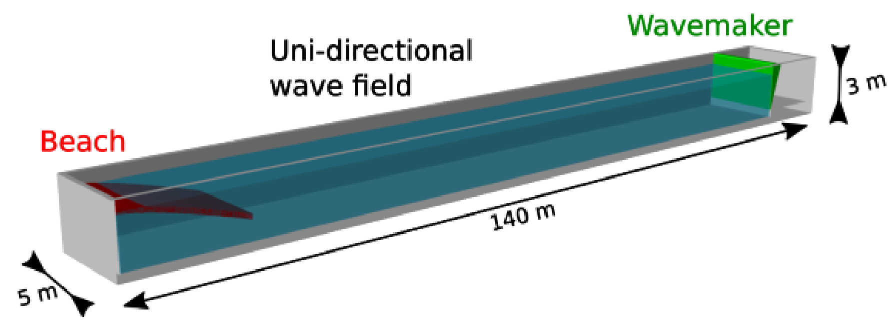


$$\xi_1 = 0.5\,i$$
$$\xi_2 = 2.4 + i$$

Collisions are elastic
(due to the integrable nature
of the NLSE)

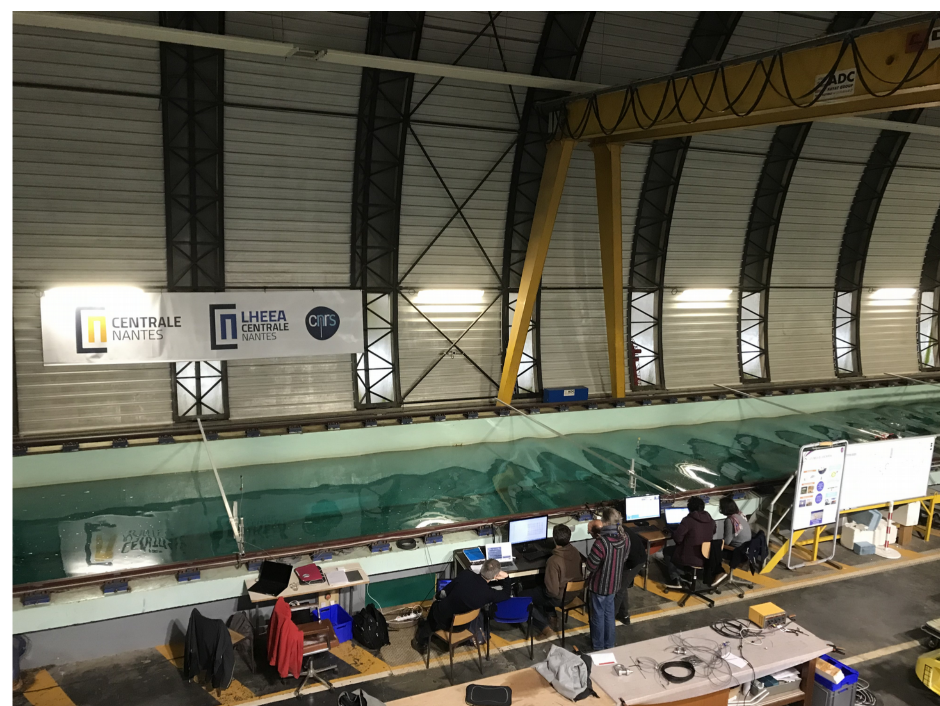
Water wave experiments

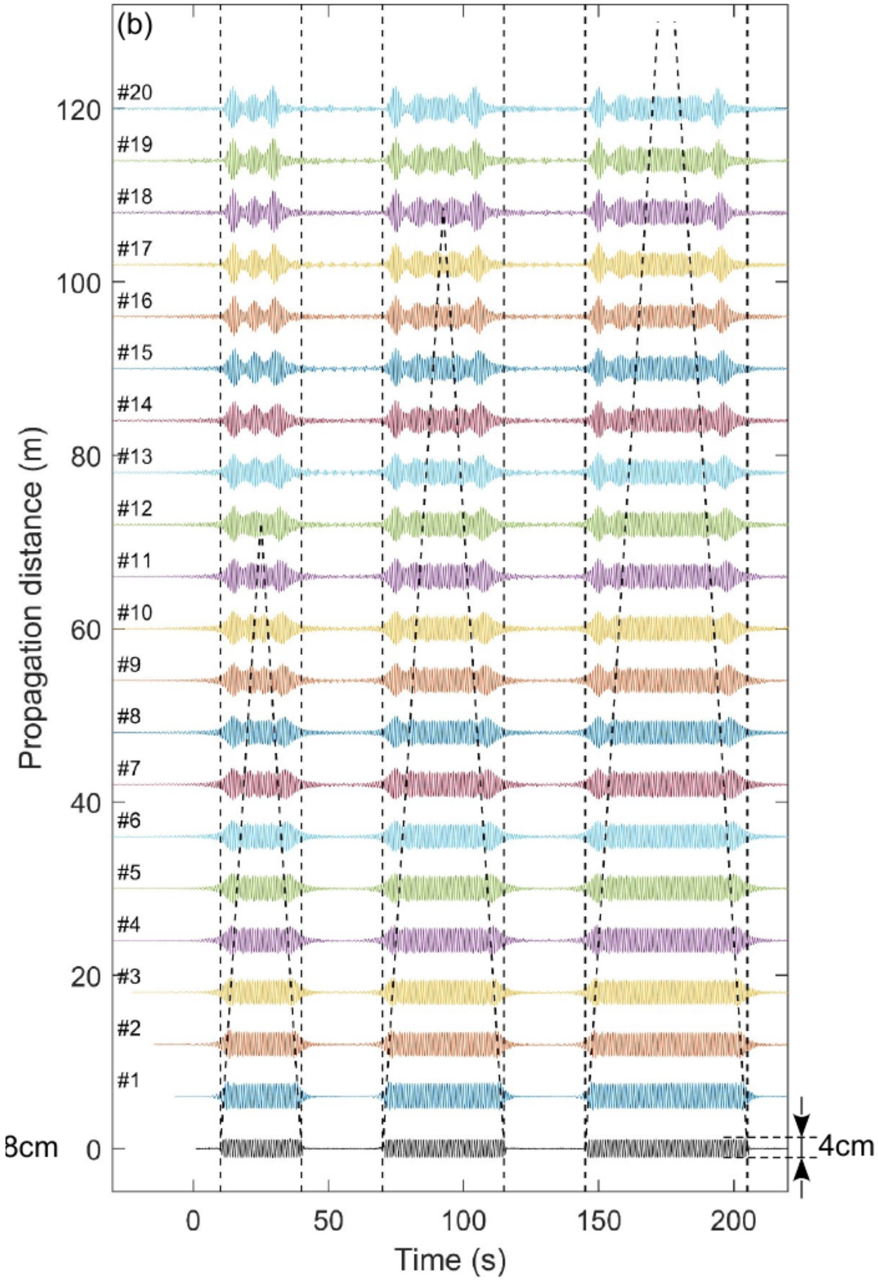




Experiments made in Ecole Centrale de Nantes (France)

Felicien Bonnefoy, Pierre Suret, Alexey Tikan, Francois Copie, Gaurav Prabhudesai, Guillaume Michel, Guillaume Ducrozet, Annette Cazaubiel, Eric Falcon, Gennady El, and Stephane Randoux





Measurement of scattering data (discrete spectrum) in water wave experiments (box problem – nonlinear diffraction)

Deep-water regime (focusing 1D-NLSE)

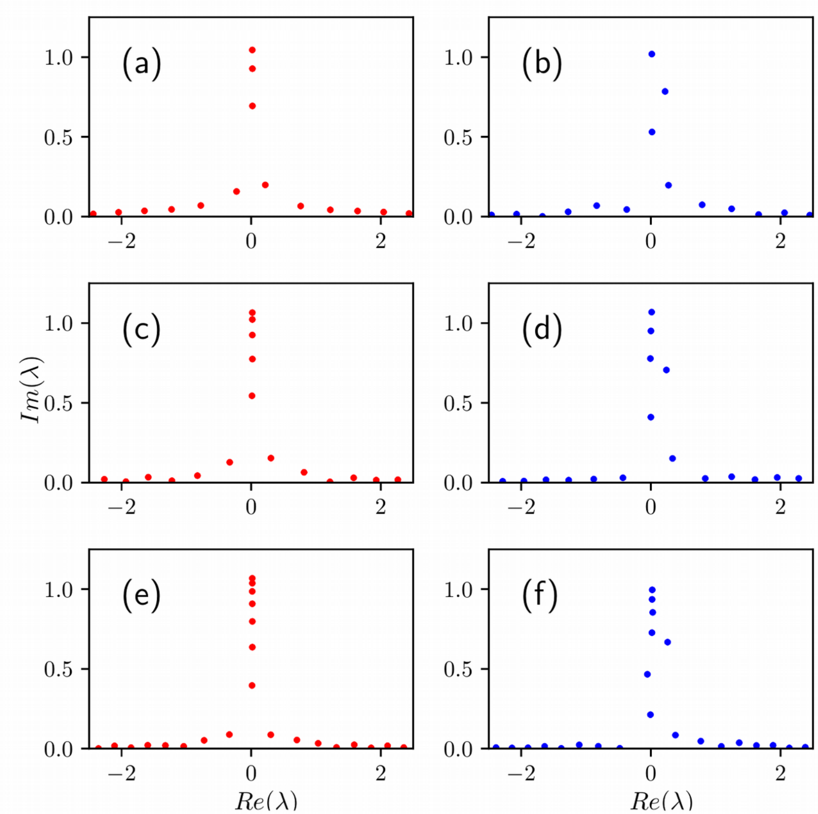
$$i\frac{\partial A}{\partial z} + \frac{k_0}{\omega_0^2}\frac{\partial^2 A}{\partial t^2} + k_0^3|A|^2 A = 0$$

$$\omega_0^2 = k_0 g$$

steepness

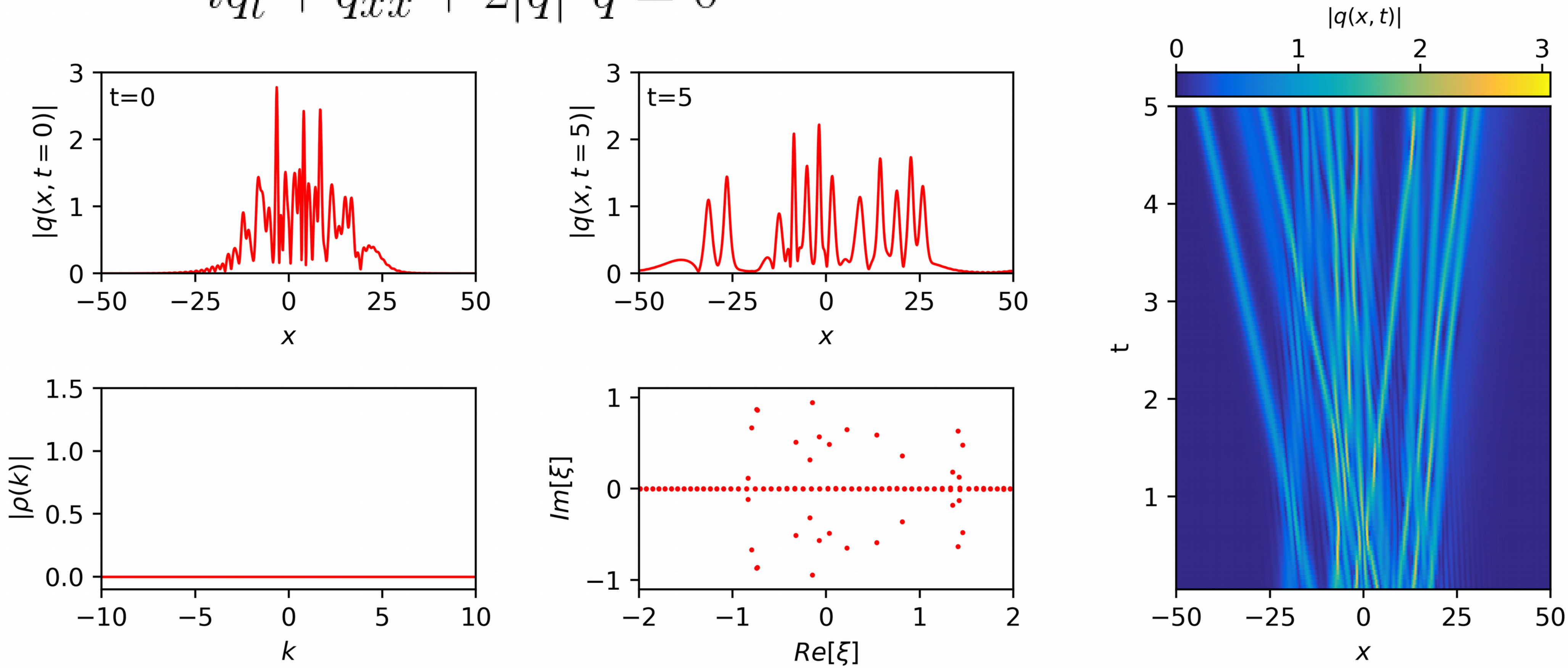
$$k_0 a = 0.082$$

Discrete Spectra (solitonic content)



From Benjamin-Feir instability to focusing dam breaks in water waves: F. Bonnefoy, A. Tikan, F. Copie, P. Suret, G. Ducrozet, G. Pradehusai, G. Michel, A. Cazaubiel, E. Falcon, G. El, S. Randoux Phys. Rev. Fluids 5, 034802 (2020)

$$iq_t + q_{xx} + 2|q|^2 q = 0$$



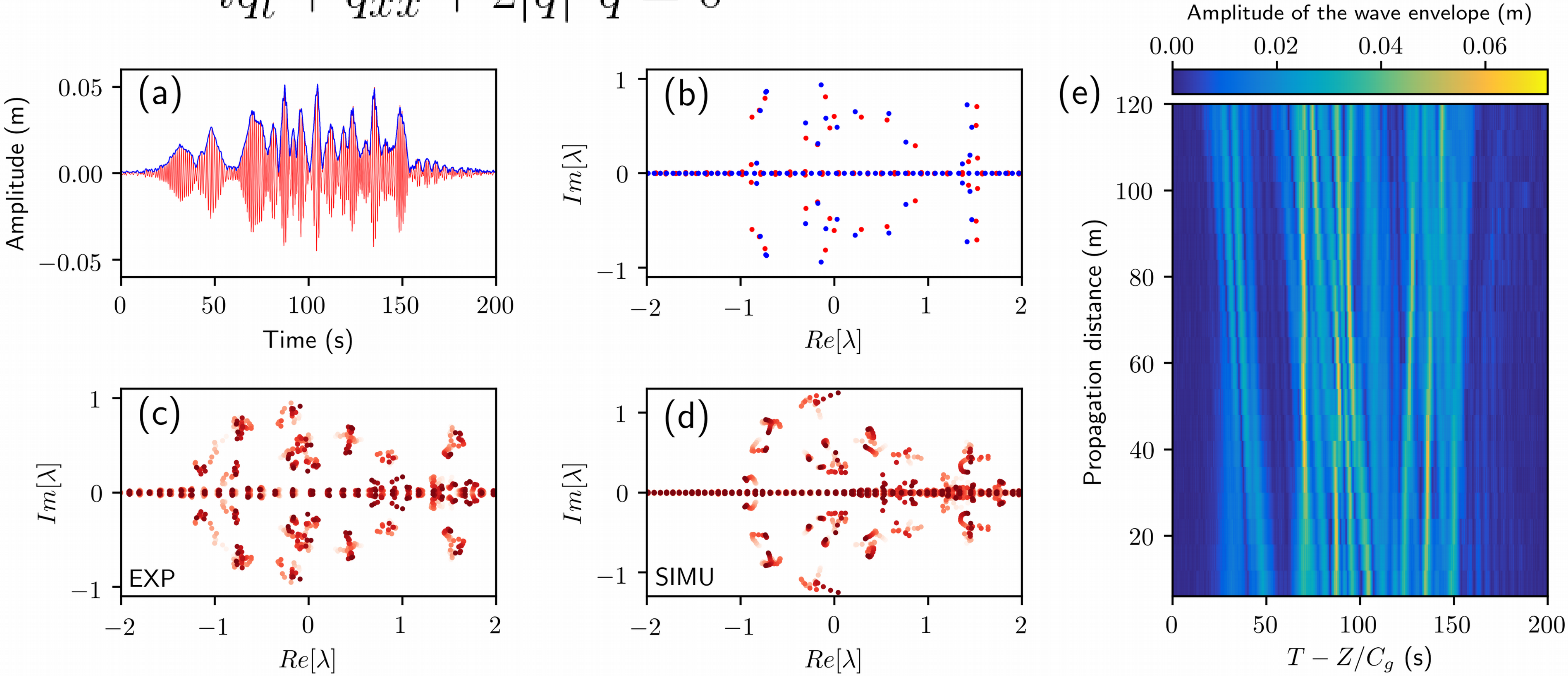
$$\xi_j \ (j = 1 \rightarrow 16)$$
$$C_j \ (j = 1 \rightarrow 16)$$

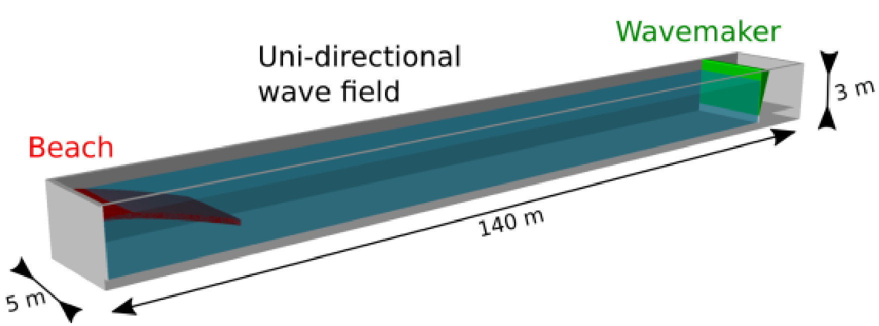
Numerical synthesis of a random ensemble of 16 NLSE solitons

Numerical methods used to build N-soliton solutions in :
Gelash, A. and Agafontsev D., Physical Review E, 98, 042210 (2018)

Experiments in a 1D water tank

$$iq_t + q_{xx} + 2|q|^2 q = 0$$



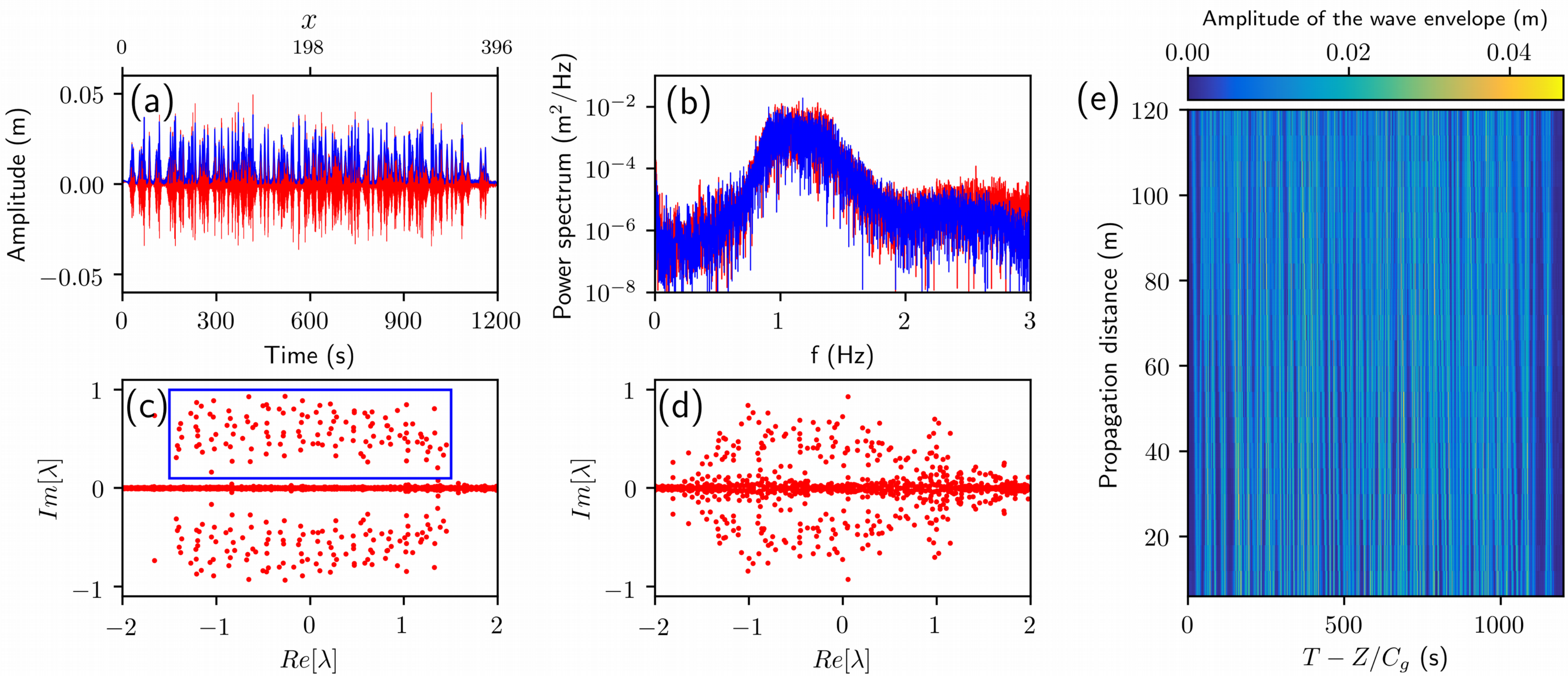


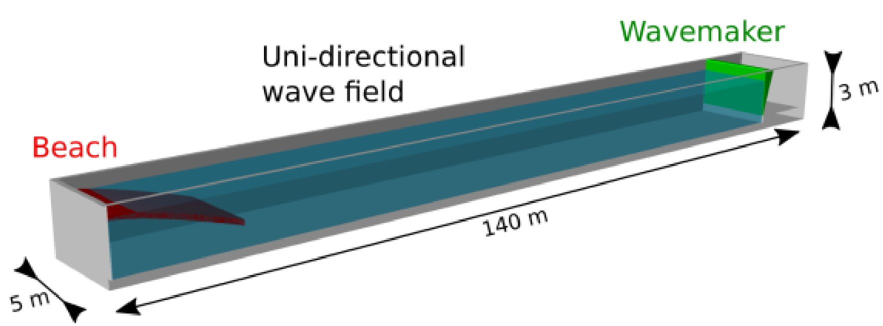
Experiments with 1D deep-water surface gravity waves: nonlinear spectral synthesis of a random ensemble of 16 solitons

Nonlinear spectral synthesis of soliton gas in deep water surface gravity waves. P. Suret et al PRL (2020)

Soliton gas

Experiments with water waves

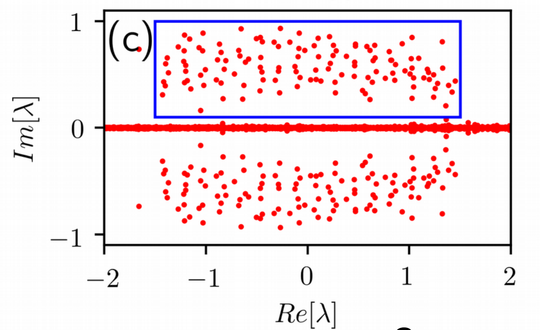




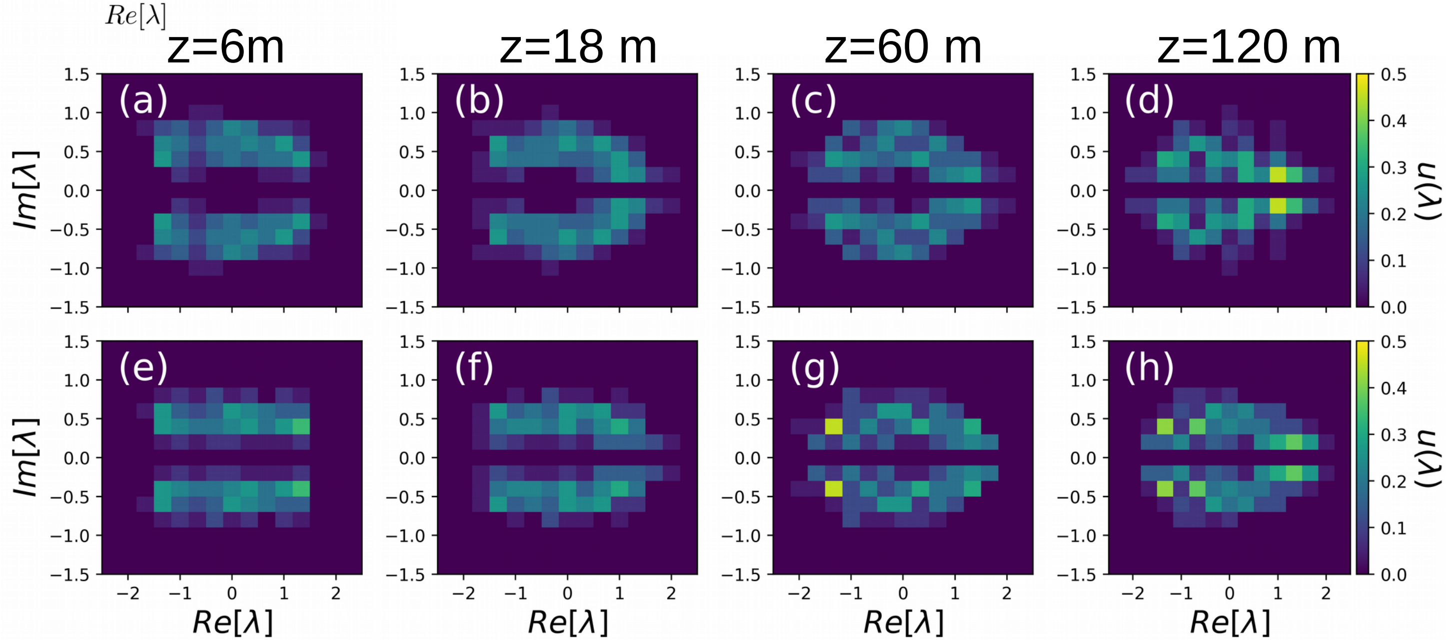
Experiments with 1D deep-water surface gravity waves: nonlinear spectral synthesis of a gas of 128 solitons

Nonlinear spectral synthesis of soliton gas in deep water surface gravity waves. P. Suret et al PRL (2020)

Nonlinear spectral synthesis of a gas of 128 solitons



Experimental measurement of the density of states $f(\lambda)$ of the hydrodynamic soliton gas



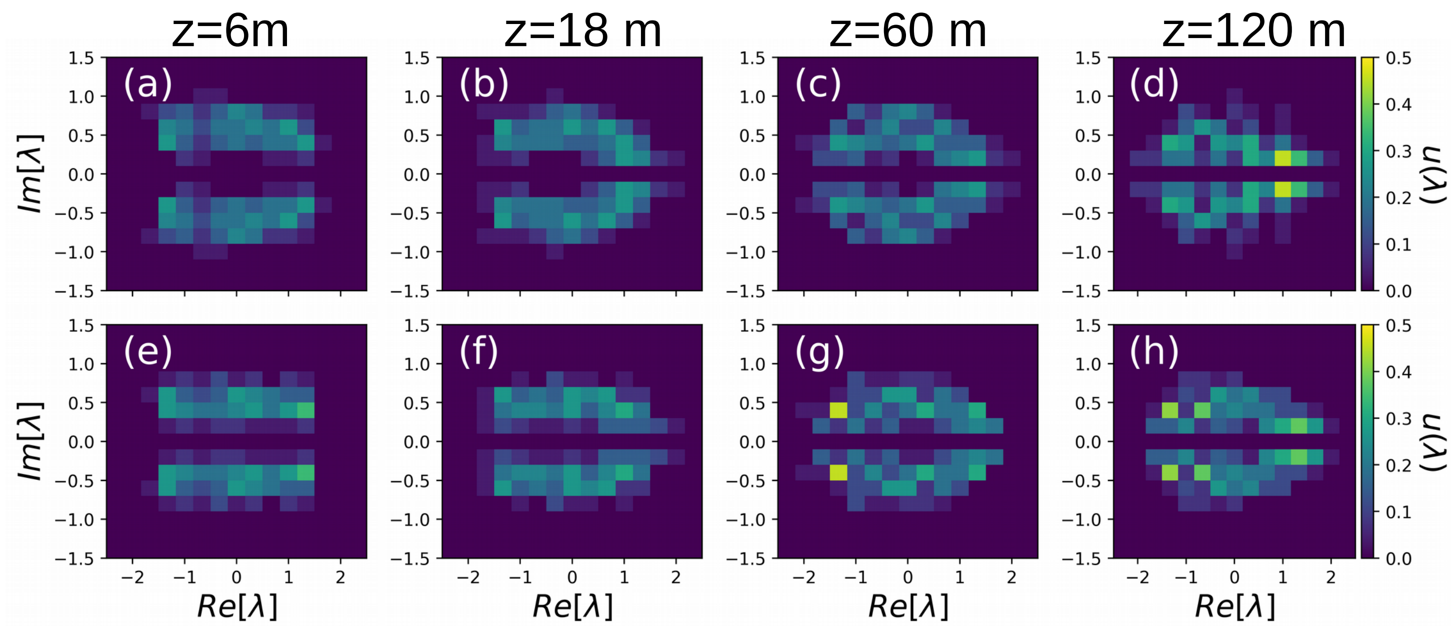
Experiments

Numerics

$$f(\lambda)d\lambda dx$$ represents the number of soliton states found between $[\lambda, \lambda + d\lambda]$ and $[x, x + dx]$

Nonlinear spectral synthesis of soliton gas in deep water surface gravity waves. P. Suret et al PRL (2020)

density of states: $f(\lambda)$



Experiments

Numerics

Kinetic equation of soliton gas

$$\begin{cases} f_t + (sf)_x = 0 \\ s(\eta) = 4\eta^2 + \dfrac{1}{\eta}\displaystyle\int_0^1 \ln\left|\dfrac{\eta+\mu}{\eta-\mu}\right| f(\mu)[s(\eta)-s(\mu)]\mathrm{d}\mu \end{cases}$$

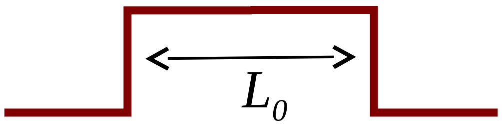
V. Zakharov, Sov. Phys. JETP **33**, 538 (1971)
G. El and A.M. Kamchatnov Phys. Rev. Lett. 95, 204101 (2005)
G. El and A. Tovbis, Phys. Rev. E **101**, 052207 (2020)

Modulation instability of a plane wave

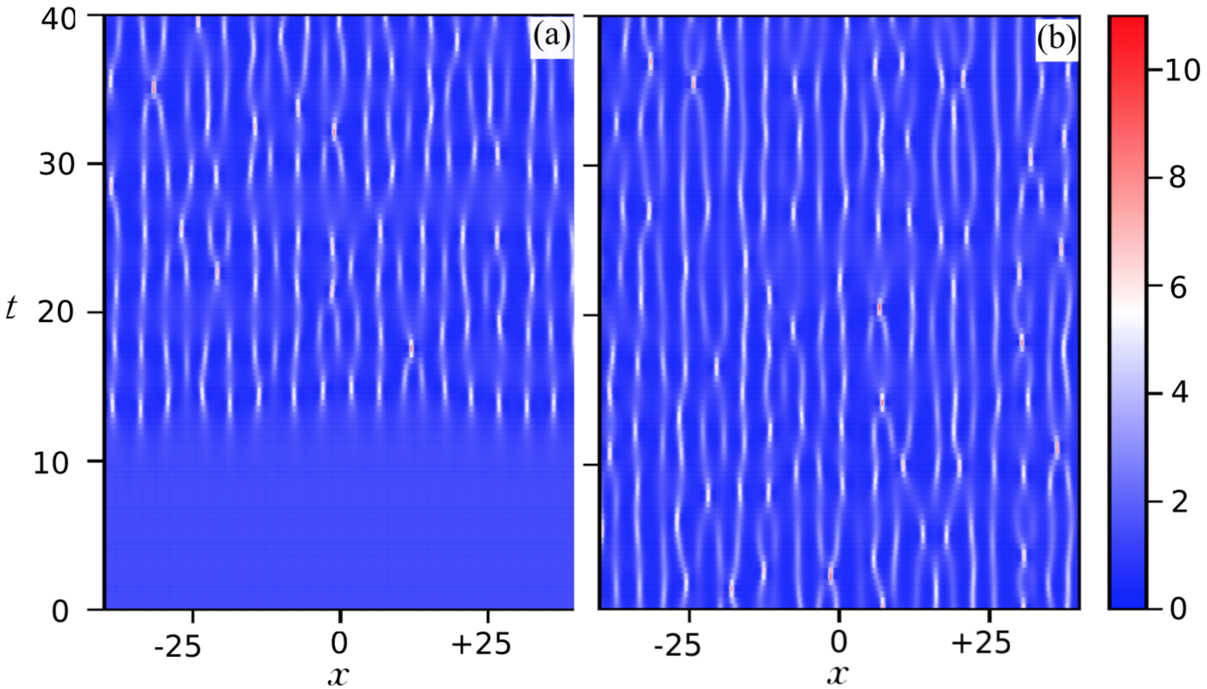
$$i\frac{\partial\psi}{\partial t}+\frac{1}{2}\frac{\partial^2\psi}{\partial x^2}+|\psi|^2\psi=0$$

$$\psi(t=0,x)=1+\eta(x)$$

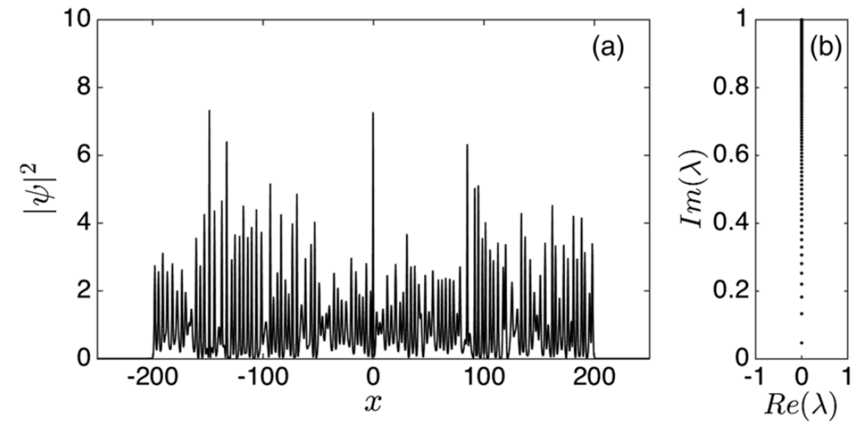
where η is the noise with $\langle\eta\rangle = 0$ and $\sqrt{\langle|\eta|^2\rangle} \ll 1$



$$\lambda_n = i\,\beta_n = i\sqrt{1-\left[\frac{\pi(n-\frac{1}{2})}{L_0}\right]^2},\quad n=1,2,\ldots,N \quad N=\mathrm{int}[L/\pi]$$



Box model with norming constants having random phases



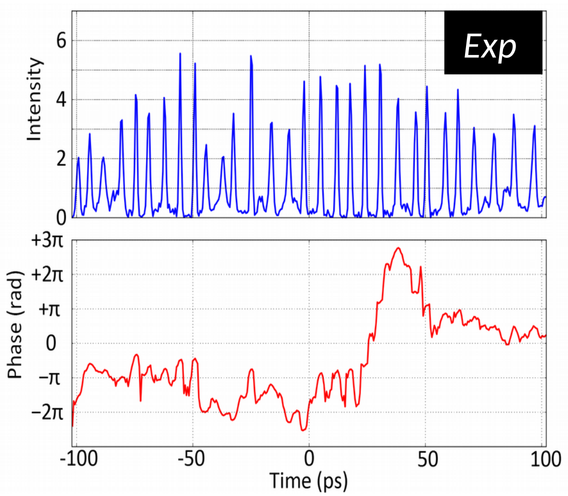
Gelash, A. , Agafontsev, D., Zakharov, V., El, G., Randoux, S., Suret, P. , Phys. Rev. Lett (2019)

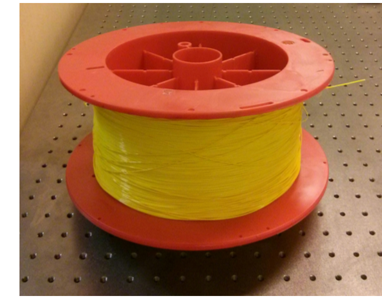
Soliton gas

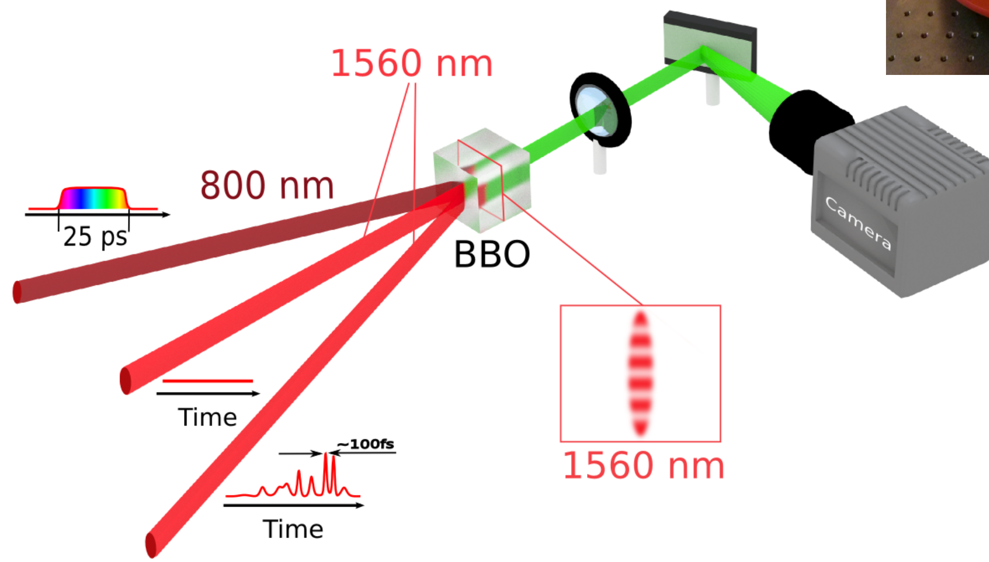
Experiments in Optics

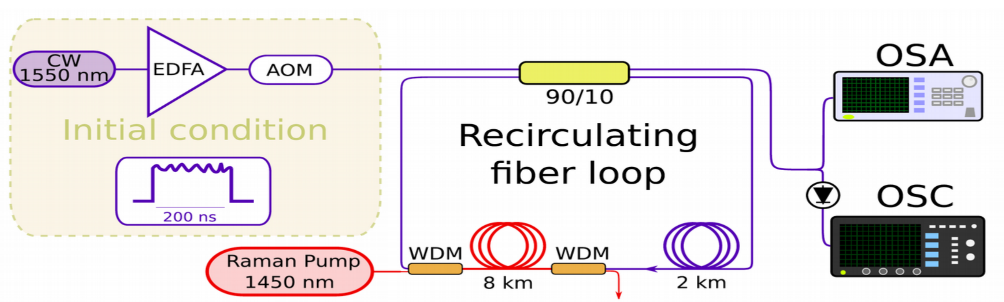
Heterodyne temporal imaging phase & amplitude (SEAHORSE)

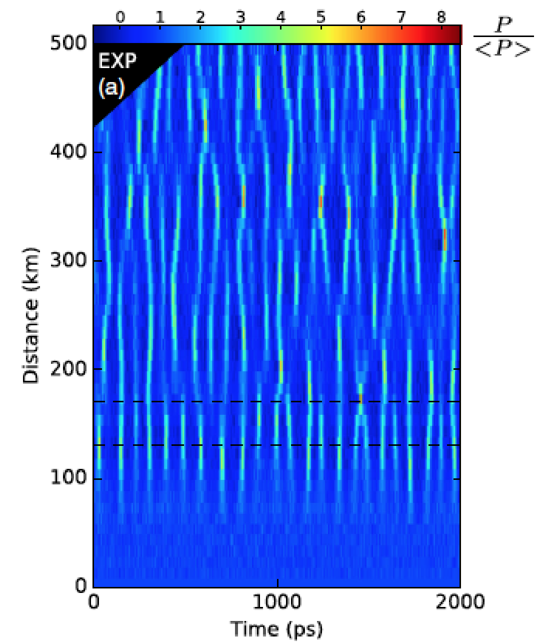
P. Suret *et al.*, Nat. Commun. **7**, 13136 (2016)
A. Tikan *et al.*, Nat. Photon. **12** (2018)
A Lebel *et al.* Opt. Lett. 46 (2), 298-301 (2021)





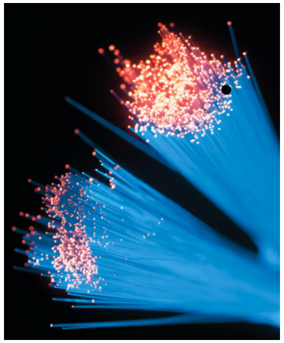




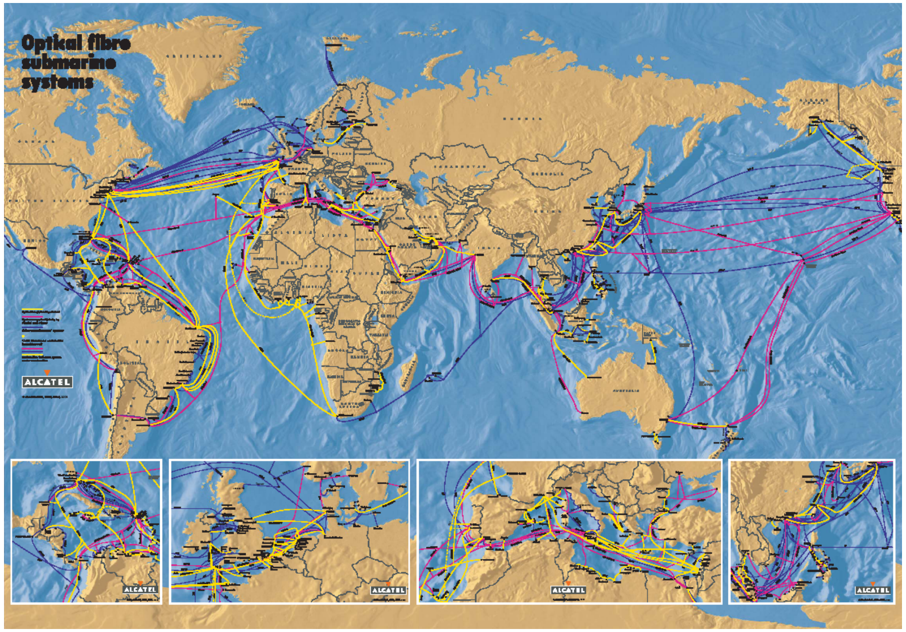


Recirculating fiber loop (PHLAM)
(single-shot space-time measurement)
PRL 123, 093902 (2020), PRL 122, 054101 (2020)

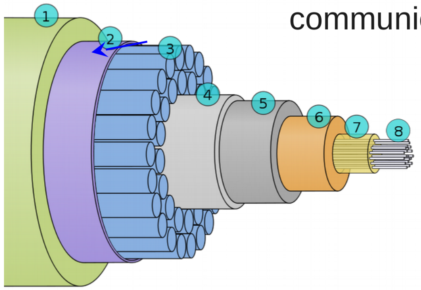
99% of global data traffic is provided by fibre communication links.



Single mode optical fibres were introduced in telecom networks only **three decades** ago. Performance increased tremendously in **30 years**: 10 Gbit/s per fibre in 1995 and multi Tbit/s systems commercially available today.

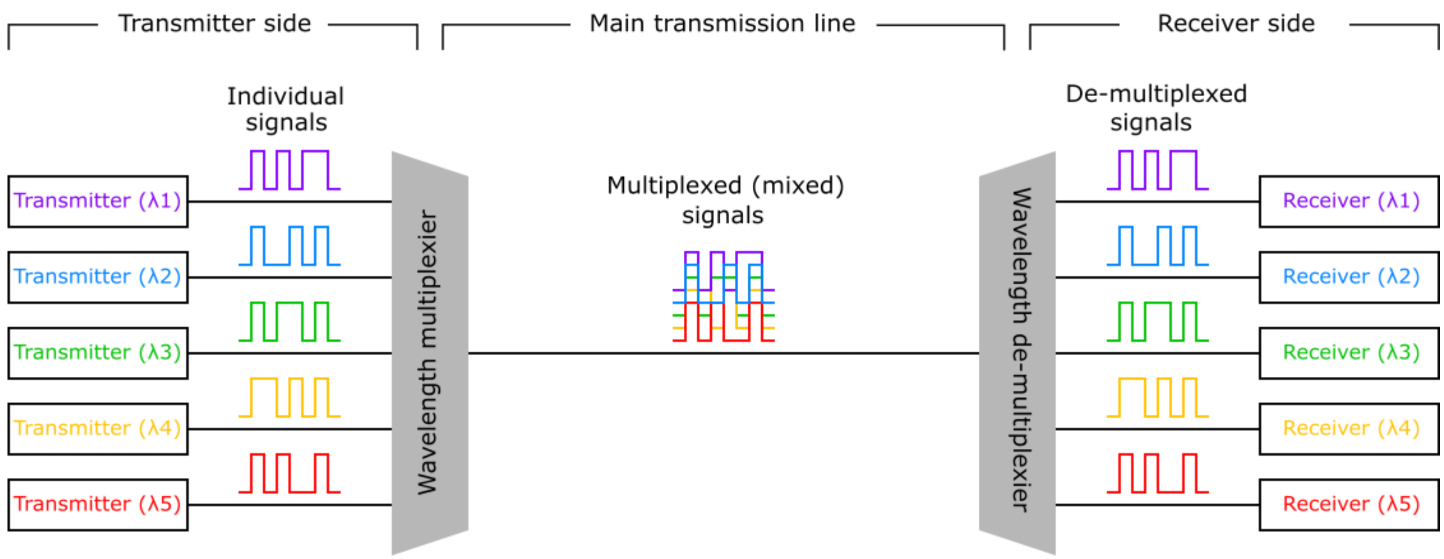


cross-section of a submarine communications cable.



- optical fibres

- Fibre-optic communication links operate according to the principle of **wavelength division multiplexing** (WDM)



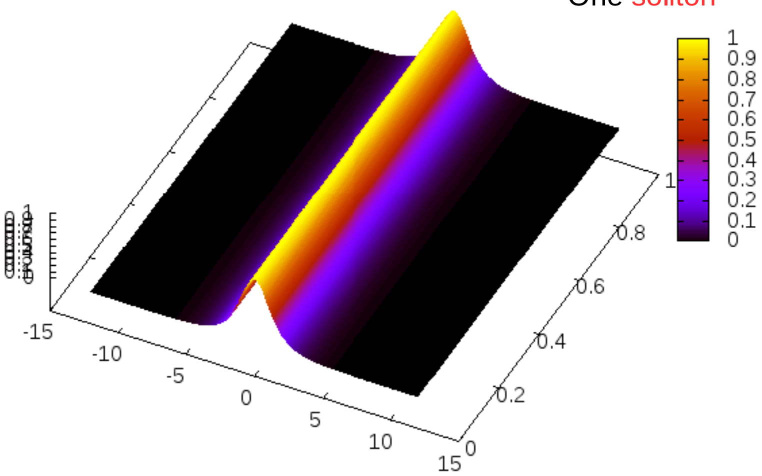
- Linear (Fourier) theory is the underlying operating principle

- Linear superposition principle is a key point

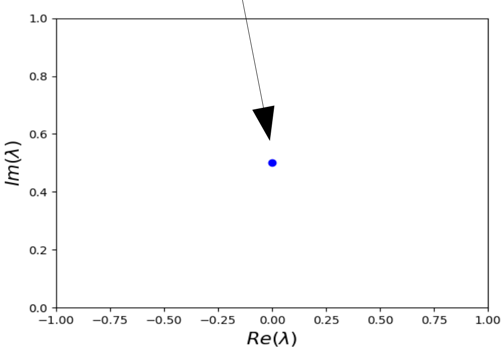
- Fiber nonlinearity is a major drawback (**broadening of the spectrum** -> cross-talk between communication channels)

- Nonlinear eigenvalue communication : fiber nonlinearity is not considered as a drawback but **it is taken to advantage**.
- **Nonlinearity and dispersion can be balanced**, resulting in **solitons** which can travel over large distances while carrying information

One soliton



One discrete eigenvalue
in the IST spectral complex plane

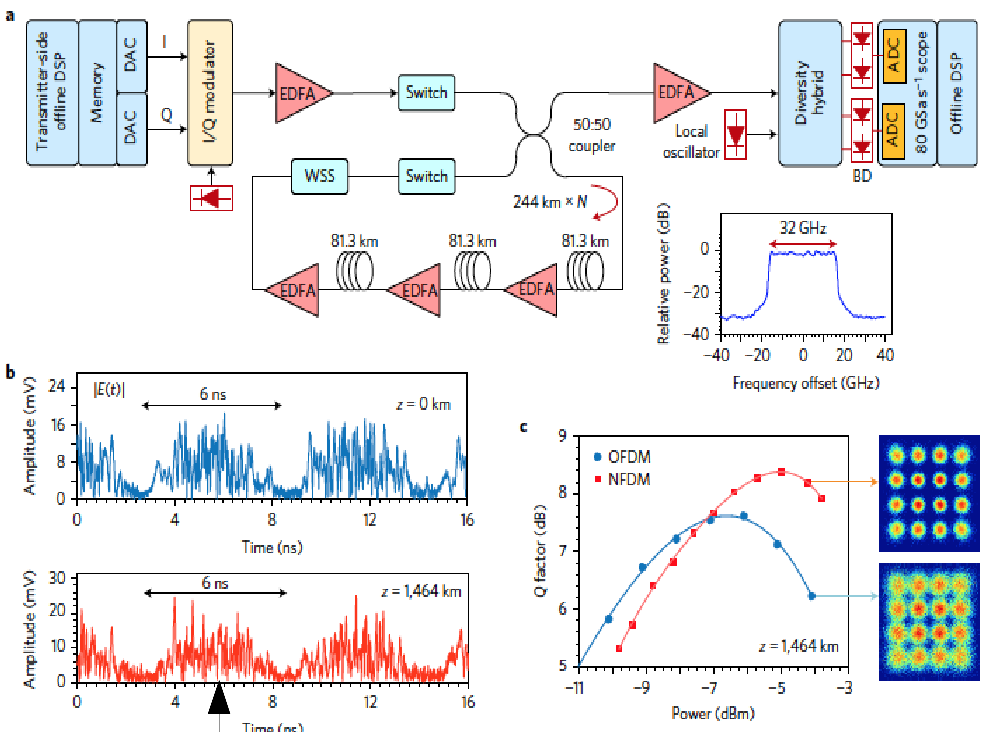


Nonlinear Schrodinger equation $$iq_t = q_{xx} + 2\sigma|q|^2 q$$

$q(x, t = 0)$ —— Nonlinear Fourier Transform / Inverse Scattering Transform (IST) ——→ $S(\xi, t = 0)$ $\left\{ \begin{array}{l} [\xi_j \, , \, C_{j,0}]_{j=1}^{N} \\ (b/a)(\xi, t = 0) \end{array} \right\}$ **Scattering data**

Nonlinear eigenvalue communication using soliton ensembles

Le et al, Nature Photonics 11, 570 (2017)



Information is “hidden” in a signal looking like random or “turbulent”

- The signal is retrieved using the IST method

- State of the art : Ensemble of 3~4 solitons are used for transmission of information in fiber links
- The generation of larger soliton ensembles is not yet mastered in fiber communication
- The field of nonlinear eigenvalue communication is connected to the field of integrable turbulence

Nonlinear Fourier transform for optical data processing and transmission: advances and perspectives.
S. Turitsyn et al, Optica **4**, 307 (2017)

Stéphane Randoux, Pierre Suret, François Copie
Alexey Tikan (PhD), Rebecca El Koussaifi (PhD), Adrien Kraych (PhD), Alexandre Lebel (PhD), Martin Dufour (PhD), Alexandre Mucci (PhD)
Christophe Szwaj, Clément Evain, Serge Bielawski
Laboratoire de Physique des Lasers, Atomes et Molécules (Phlam), Univ. de Lille, France

Gennady El, Thibault Congy, Giacomo Roberti (PhD), *Northumbria univ., UK*
Andrey Gelash, *Novosibirsk, Russia*
Dmitry Agafontsev, *Moscow, Russia*
Vladimir Zakharov, *Landau Institute for Theoretical Physics, Chernogolovka, Russia*
Alexander Tovbis , *Univ of central Florida, USA*

Miguel Onorato, *Torino, Italy*
Eric Falcon, Annette Cazaubiel (PhD) MSC, *Univ. Paris Diderot, France*
Guillaume Michel, Gaurav Prabhudesai (ENS, PhD), *Ecole Norm. Sup., France*
Félicien Bonnefoy, Guillaume Ducrozet, *Ecole Centrale de Nantes*
Amin Chabchoub, *Univ. Of Sydney, Australia*
Nicolas Mordant, Eric Barthélémy, Hervé Michalet *Univ. Grenoble Alpes*