# Nonintersecting Brownian bridges in the flat-to-flat geometry

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in collaboration with

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- Grégory Schehr (LPTHE, Sorbonne Université)

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# How to simulate a Brownian motion?

- Let us start with the simple Brownian motion

$$\frac{dx(t)}{dt} = \sqrt{2\,D}\,\eta(t) \quad , \quad x(0) = a$$

Gaussian white noise with

$$\langle \eta(t) \rangle = 0 \quad , \quad \langle \eta(t)\eta(t') \rangle = \delta(t - t')$$

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and $D$ is the diffusion constant

- To simulate it numerically we discretize time with increments $\Delta t \ll 1$

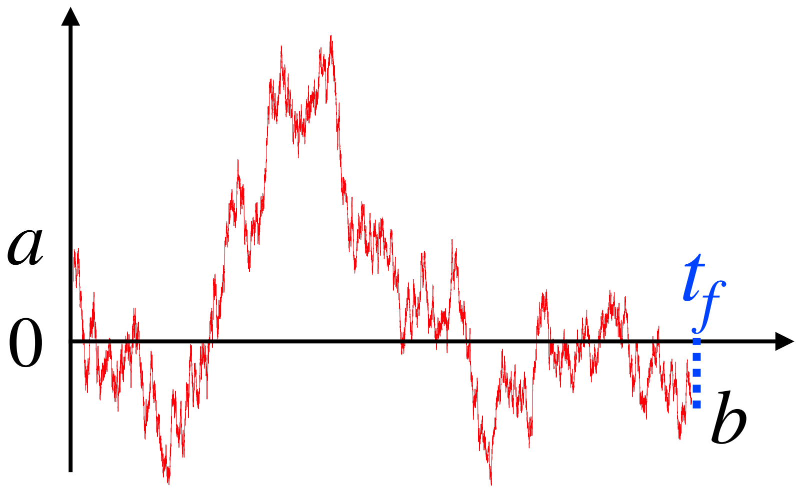
$$x(0) = a$$

$$x(n\,\Delta t) = x((n-1)\Delta t) + \sqrt{2D}\,\eta(t)\,\Delta t \quad , \quad n = 1,2,\cdots$$

Gaussian random variable with zero mean and variance $2D\,\Delta t$

# How to simulate a Brownian bridge?

- A Brownian bridge is a Brownian motion, starting from $x_B(0) = a$ and conditioned to end at $x_B(t_f) = b$



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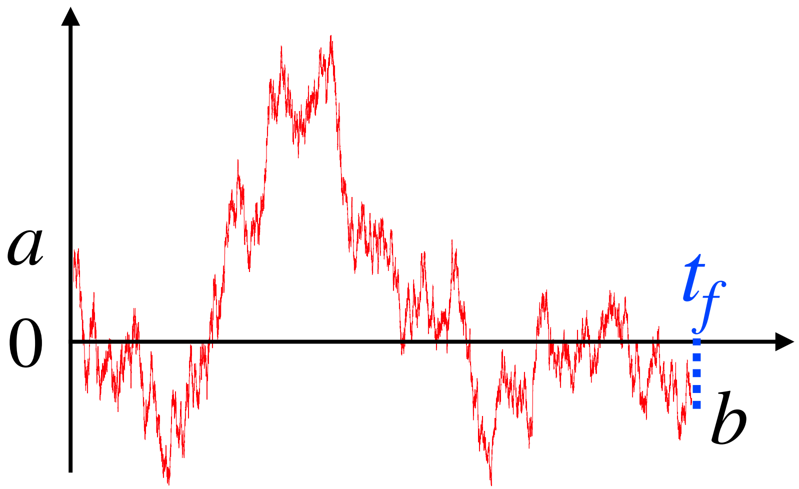


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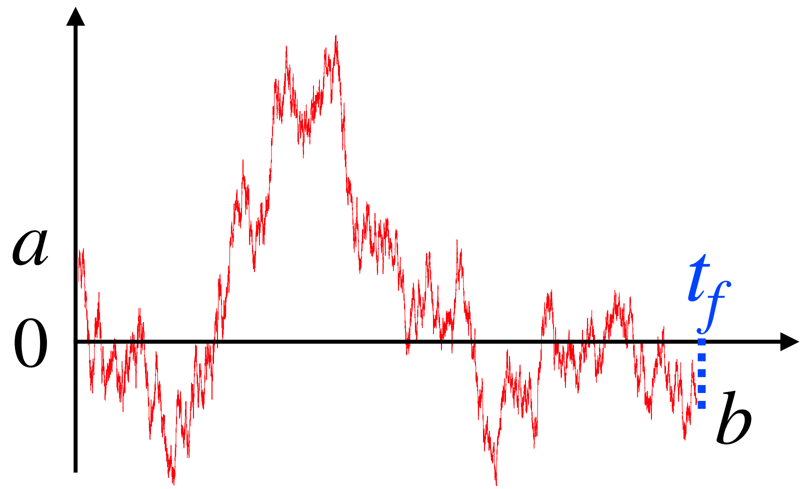


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- There exist various efficient ways to simulate a Brownian bridge, e.g.

$$x_B(t) = x(t) + \frac{t}{t_f}(b - x(t_f))$$

where $x(t)$ is a std Brownian motion starting at $x(0) = a$

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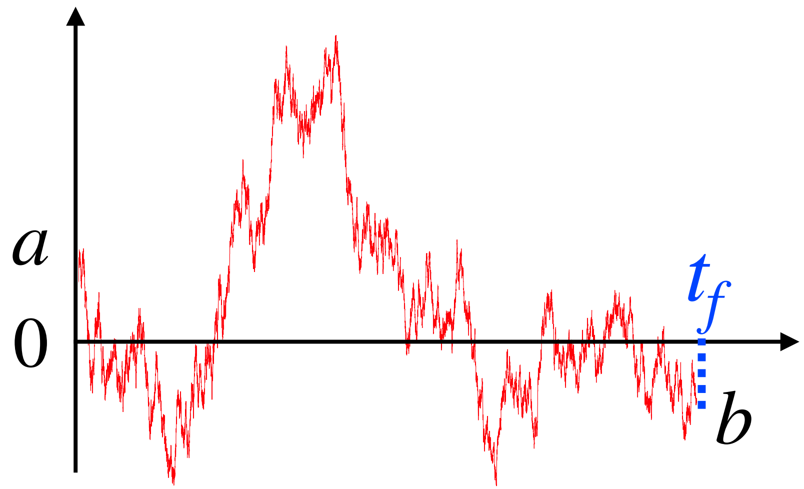


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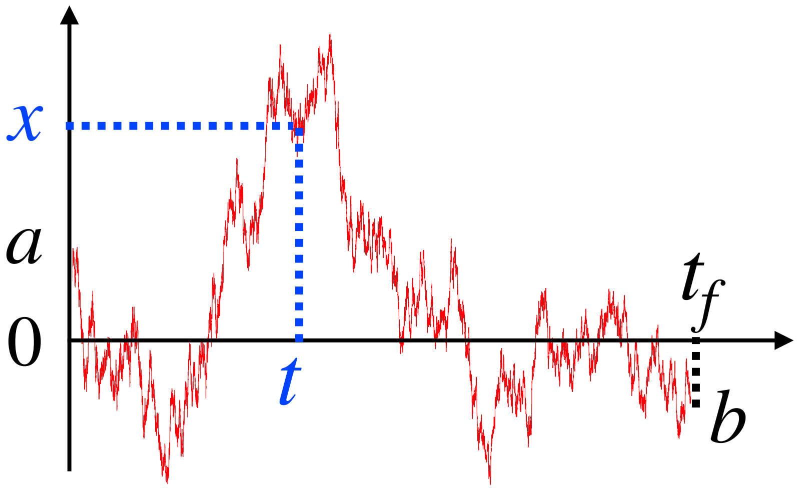
- One can also write an effective Langevin equation

Chetrite, Touchette (2015),
S. N. M., Orland (2015)

$$\frac{dx_B(t)}{dt} = \frac{b - x_B(t)}{t_f - t} + \sqrt{2D}\,\eta(t) \quad , \quad x_B(0) = a$$

# Effective Langevin Eq. for a Brownian bridge




$$\overbrace{P_{\rm BB}(x,t\,|\,b,a,t_f)}^{\tilde{P}} = \frac{\overbrace{P(x,t\,|\,a,0)}^{P}\;\overbrace{P(b,t_f\,|\,x,t)}^{Q}}{P(b,t_f\,|\,a,0)}$$

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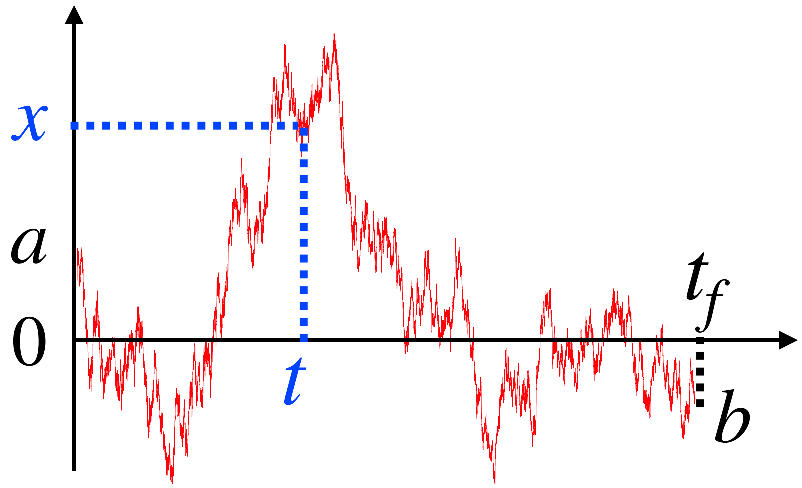


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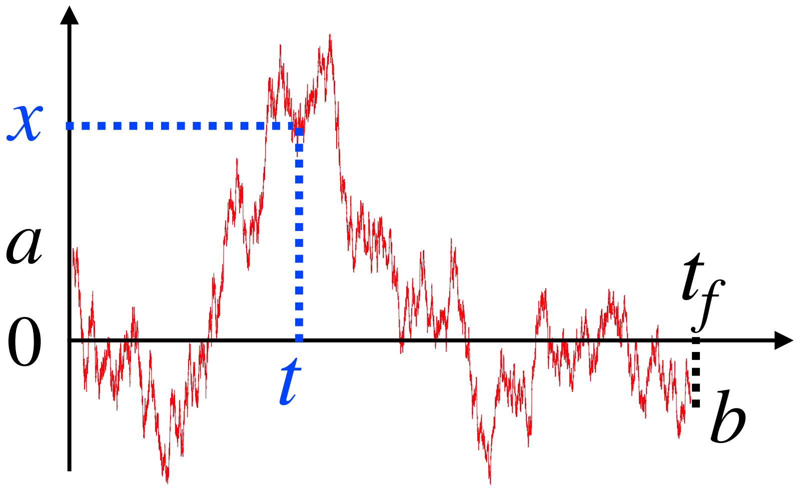


$\tilde{P}$ $\quad P \quad Q$

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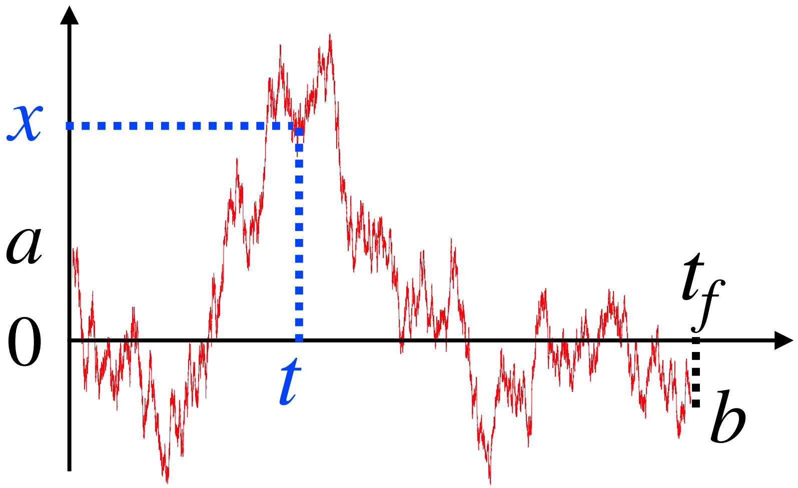


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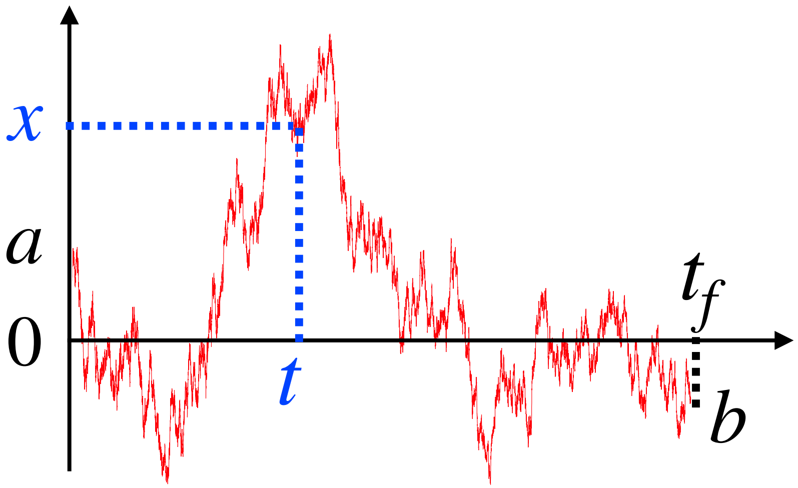


$\tilde{P}$ $\quad P \quad Q$

$$P_{\mathrm{BB}}(x, t \,|\, b, a, t_f) = \frac{P(x, t \,|\, a, 0)\, P(b, t_f \,|\, x, t)}{P(b, t_f \,|\, a, 0)}$$

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- The product $\tilde{P} = P\, Q$ satisfies $\partial_t \tilde{P} = D \partial_x^2 \tilde{P} - 2\, D\, \partial_x \left( \tilde{P}\, \partial_x \ln Q \right)$

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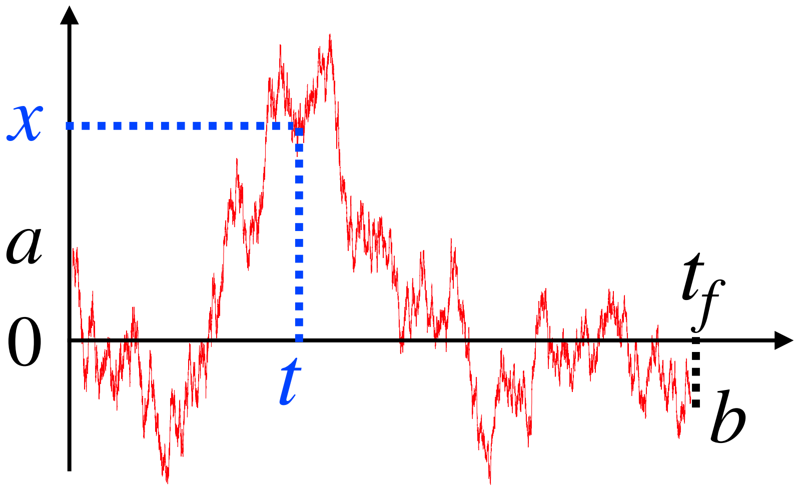


$\tilde{P}$ $\qquad\qquad$ $P$ $\qquad\qquad$ $Q$

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⇒ This corresponds to the effective Langevin Eq.

$$\frac{dx_B}{dt} = 2D\partial_x \ln Q + \sqrt{2D}\,\eta(t) \qquad \text{with } Q(b,t_f\,|\,x_B,t) = \frac{e^{-\frac{(x_B-b)^2}{4D(t_f-t)}}}{\sqrt{4\pi D(t_f-t)}}$$

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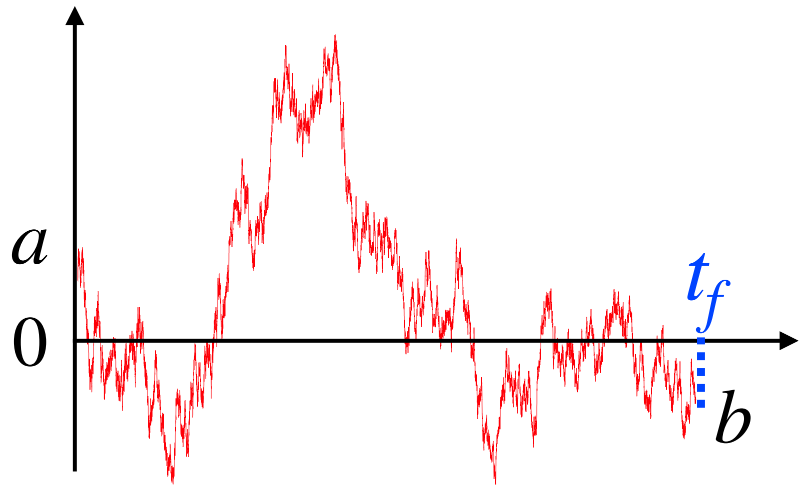


$$\frac{dx_B(t)}{dt} = \frac{b - x_B(t)}{t_f - t} + \sqrt{2D}\,\eta(t) \quad , \quad x_B(0) = a$$

Discretizing in time $\Longrightarrow$ generates a Brownian bridge trajectory in a rejection free way

# Our main motivation

Q: is it possible to generalise the effective Langevin approach from a single Brownian bridge to multiple Brownian bridges with interaction between them?

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Karlin & McGregor (1959), de Gennes (1968), Fisher (1984), Huse & Fisher (1984), Krattenthaler, Guttmann, Viennot (2000), Johansson (2002), Prähofer & Spohn (2002), Bonichon & Mosbah (2003), Katori & Tanemura (2004), Tracy and Widom (2007), Schehr, S. N. M., Comtet & Randon-Furling (2008), Borodin, Ferrari, Prähofer, Sasamoto & Warren (2009), Nadal & S. N. M. (2009), Forrester, S. N. M. & Schehr (2011), Bun, Bouchaud, S. N. M. & Potters (2014)...

THE JOURNAL OF CHEMICAL PHYSICS VOLUME 48, NUMBER 5 1 MARCH 1968

# Soluble Model for Fibrous Structures with Steric Constraints

P.-G. DE GENNES

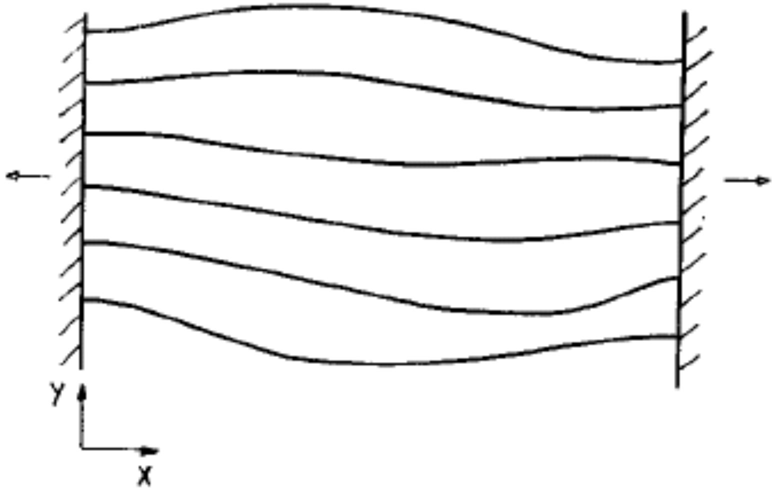
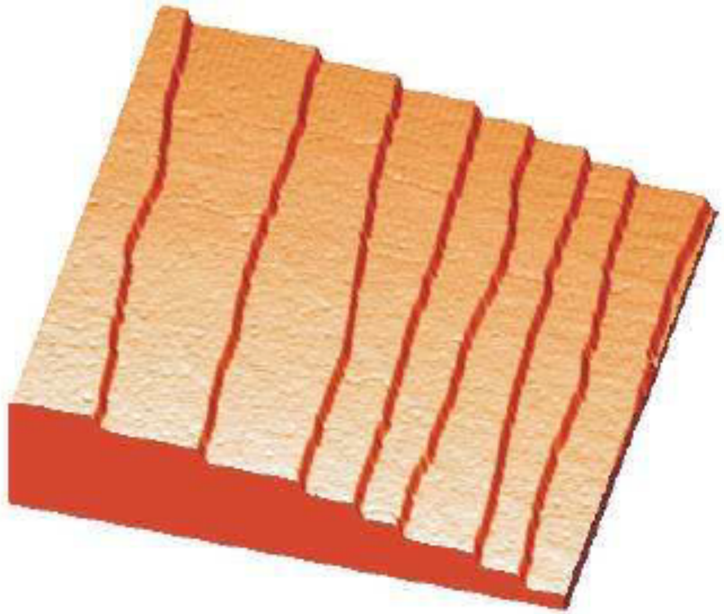
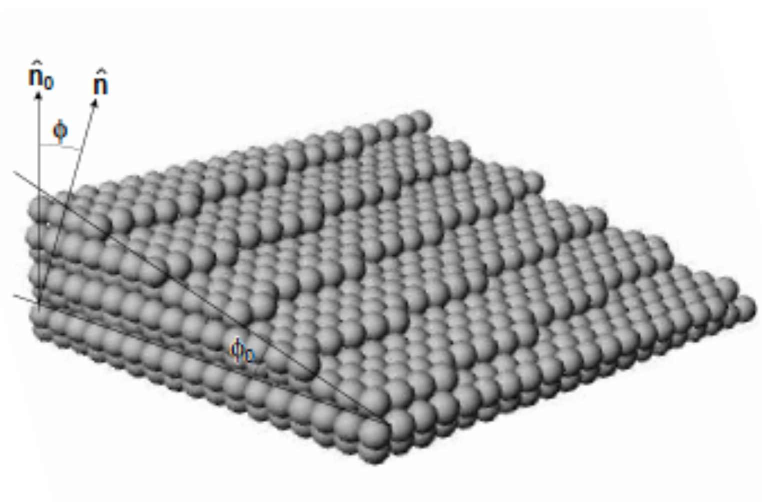
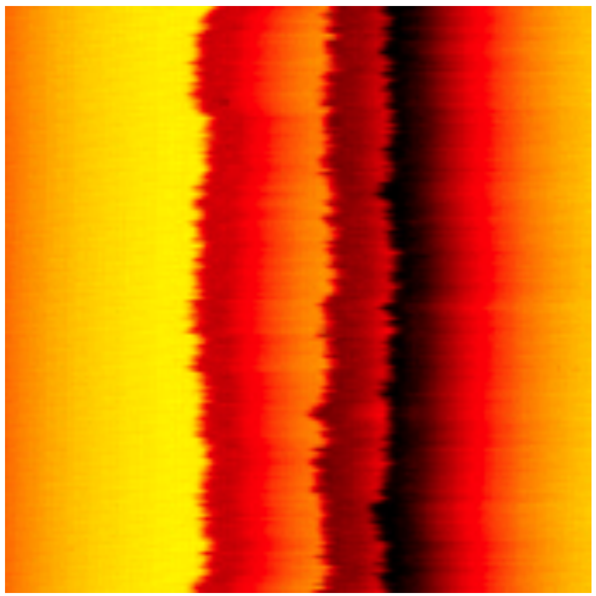



FIG. 1. Model for a two-dimensional fiber structure. The component chains are assumed to be attached to two plates I and F and placed under tension. The chains are bent by thermal fluctuations. Different chains cannot intersect each other.

# Step fluctuations on vicinal surfaces





Maryland group (2003)

# Applications in combinatorics/computer science

## Watermelon uniform random generation with applications

Nicolas Bonichon*, Mohamed Mosbah

*LaBRI-Université Bordeaux 1, 351 Cours de la Libération, 33405 Talence, France*

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- An algorithm to generate discrete time nonintersecting lattice bridges

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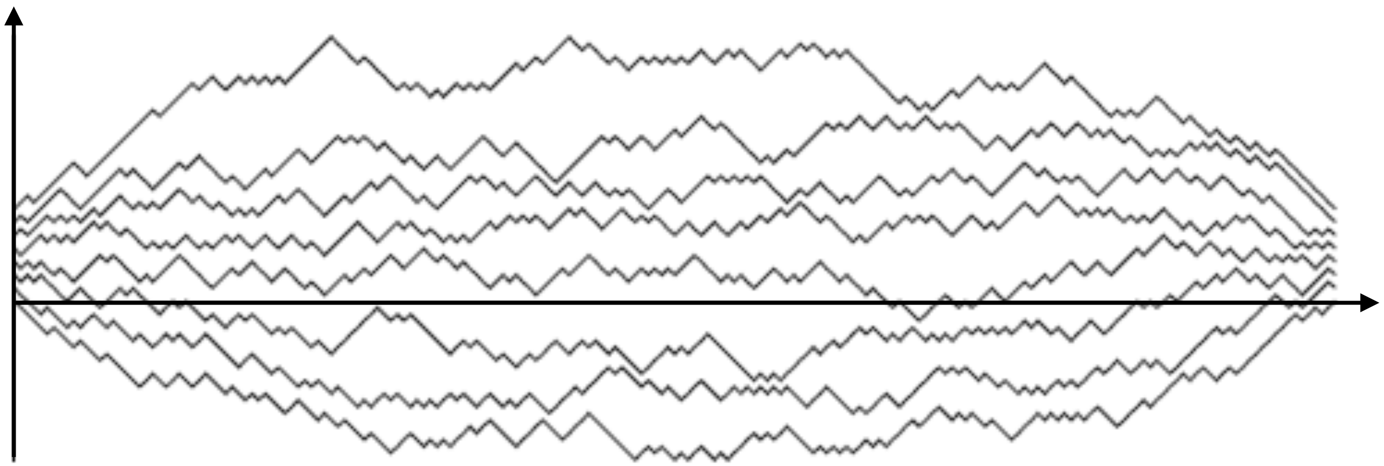
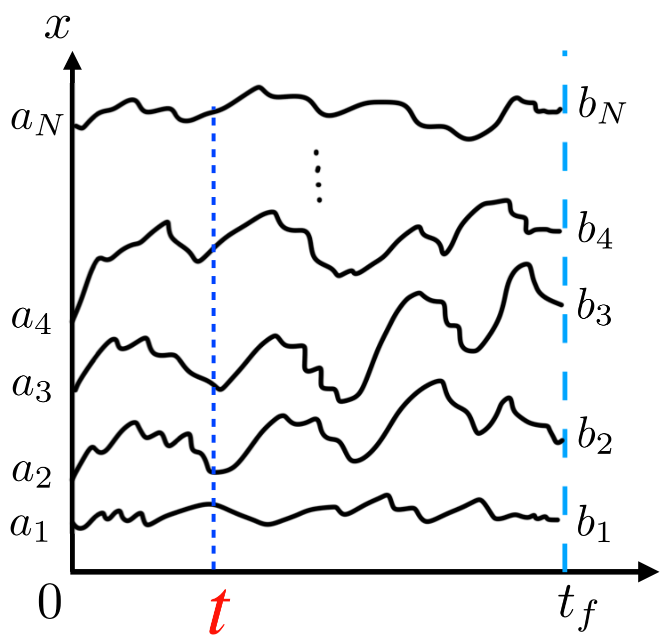


# Nonintersecting (vicious) Brownian bridges




for $a_i = b_i = \dfrac{i-1}{N}, i = 1,2,\cdots,N$

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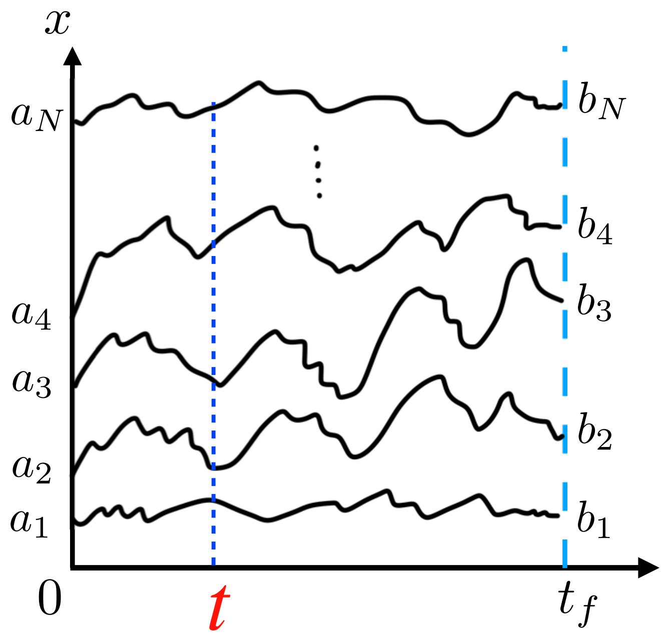


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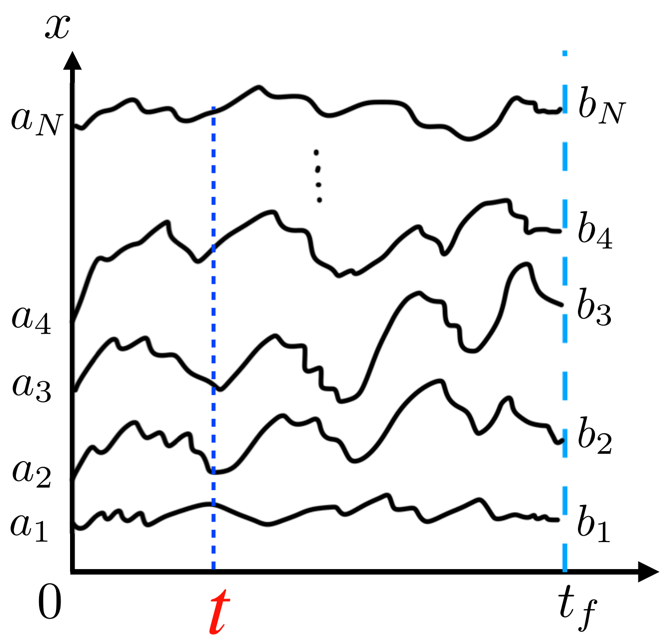


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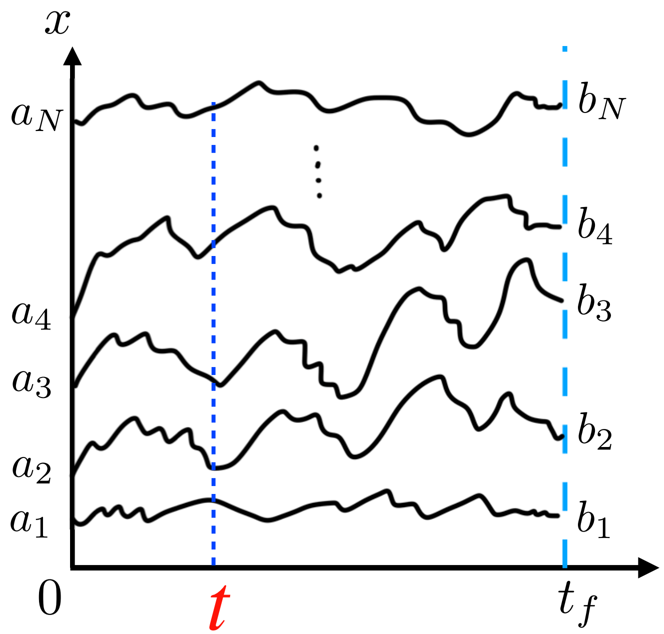


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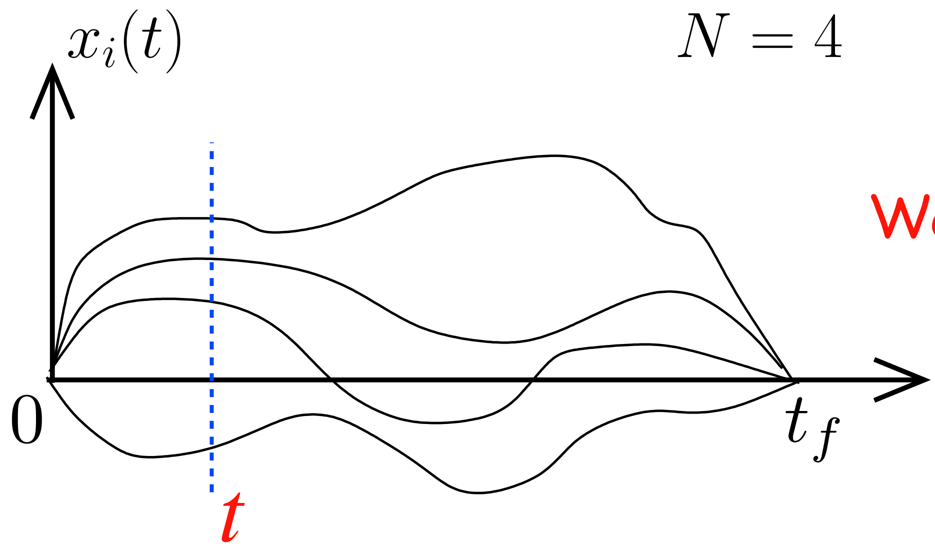
- A special case $a_i = b_i = 0$ ⟹ ``watermelons”

# "Watermelons" and random matrix theory

- Vicious bridges with $a_i = b_i = 0$ for all $i = 1, 2, \cdots, N$




Watermelon configuration

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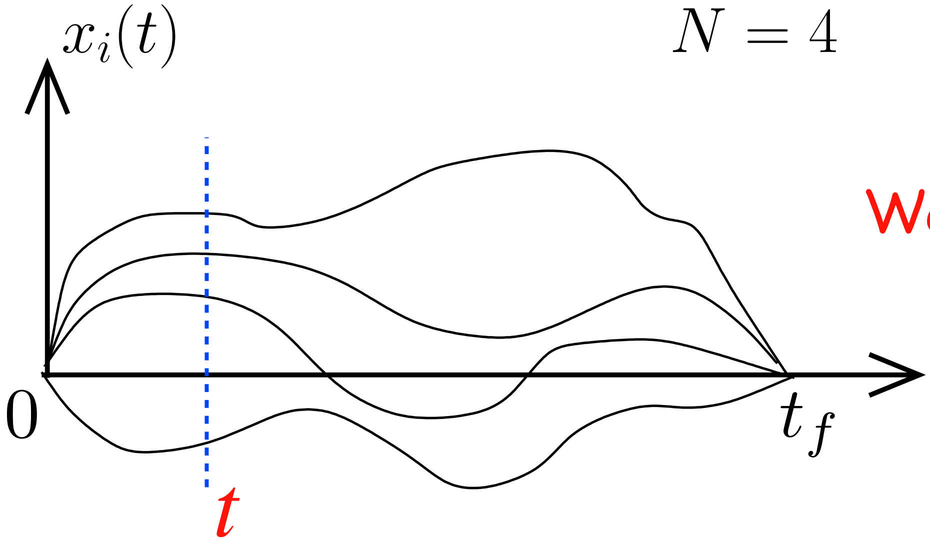


Watermelon configuration

- Joint proba. density function of $x_1(t), x_2(t), \cdots, x_N(t)$ for $0 \leq t \leq t_f$

$$P_{\rm joint}(x_1, x_2, \cdots, x_N) \propto e^{-\sum_{i=1}^N \frac{x_i^2}{2\sigma^2(t)}} \prod_{i<j}^N (x_i - x_j)^2 \quad , \quad \sigma^2(t) = \frac{2\,D\,t(t_f - t)}{t_f}$$

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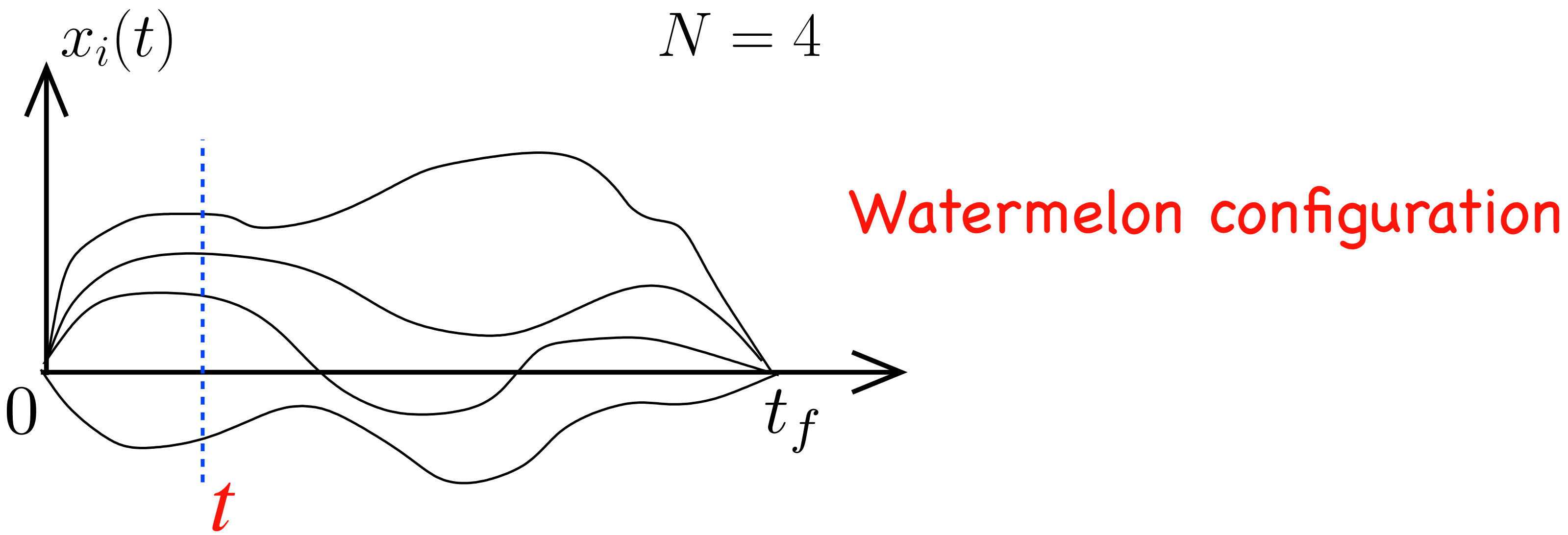


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The rescaled positions $\frac{x_i}{\sigma(t)}$ behave like the eigenvalues of the Gaussian Unitary Ensemble ($\beta = 2$) of random matrices

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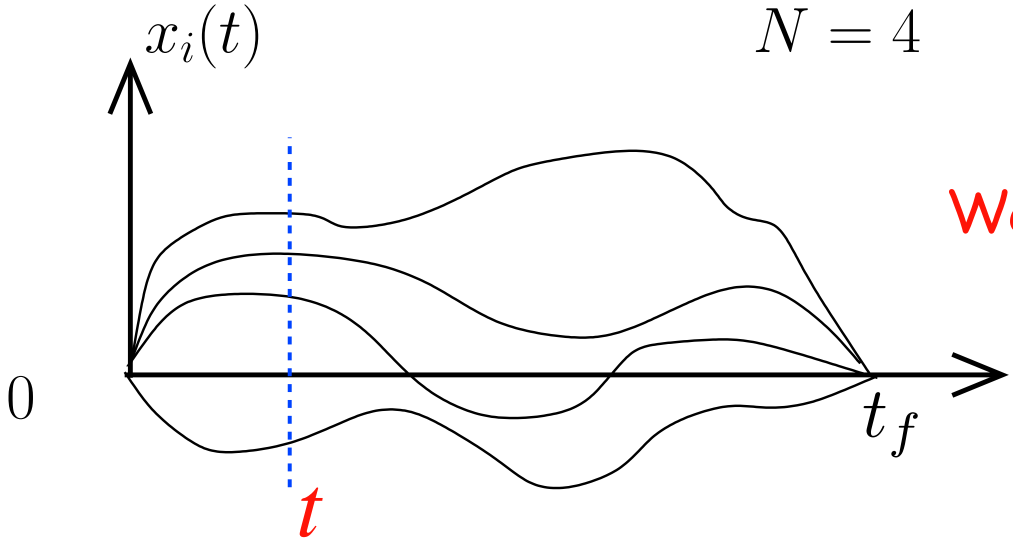


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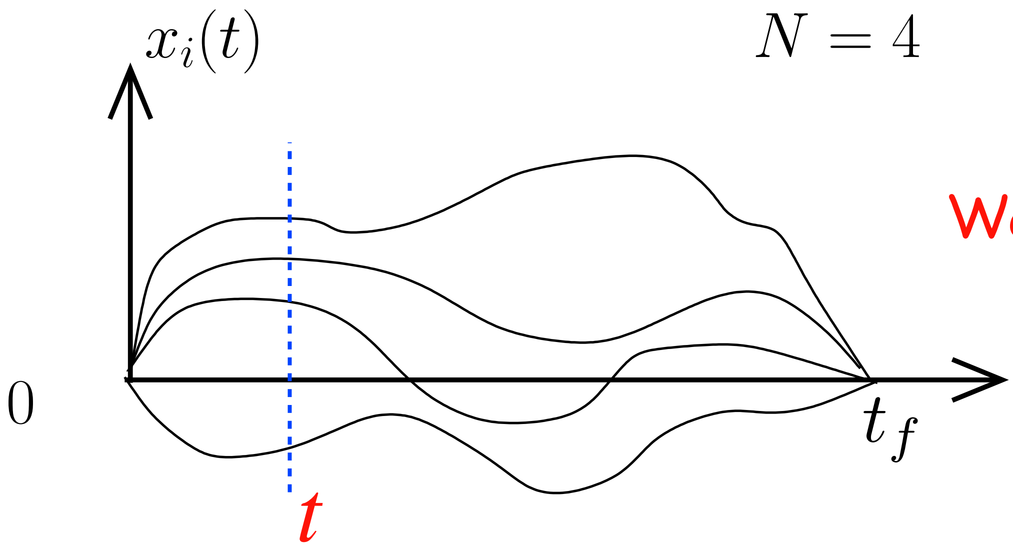


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- The average density at any intermediate $0 \leq t \leq t_f$ for large $N$

$$\rho_N(x;t) \simeq \frac{1}{\pi\sqrt{N}\sigma(t)}\sqrt{2 - \frac{x^2}{N\,\sigma^2(t)}}\ , \qquad -\sqrt{2N}\,\sigma(t) \leq x \leq \sqrt{2N}\,\sigma(t)$$

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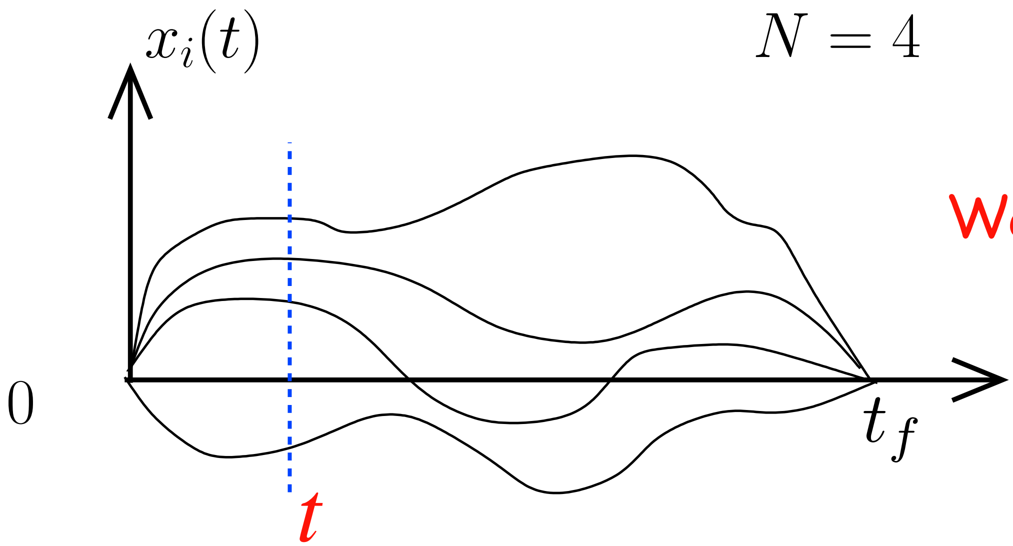


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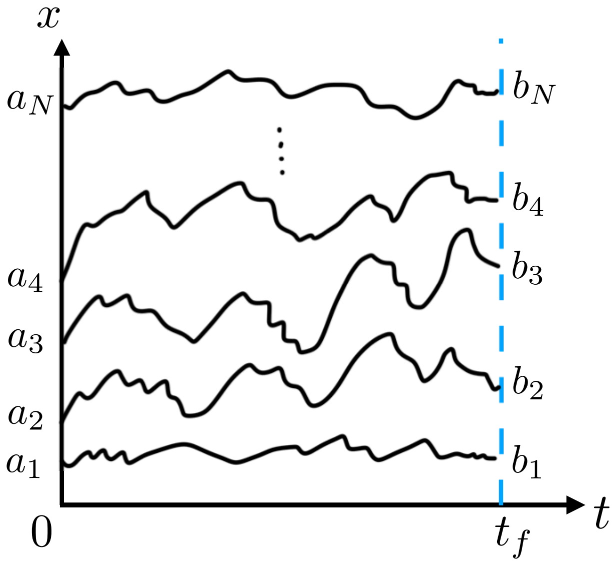
⇨ Wigner 's semi-circle for all $0 \leq t \leq t_f$

# Main results

Grela, S. N. M., Schehr (2021)

Vicious BB



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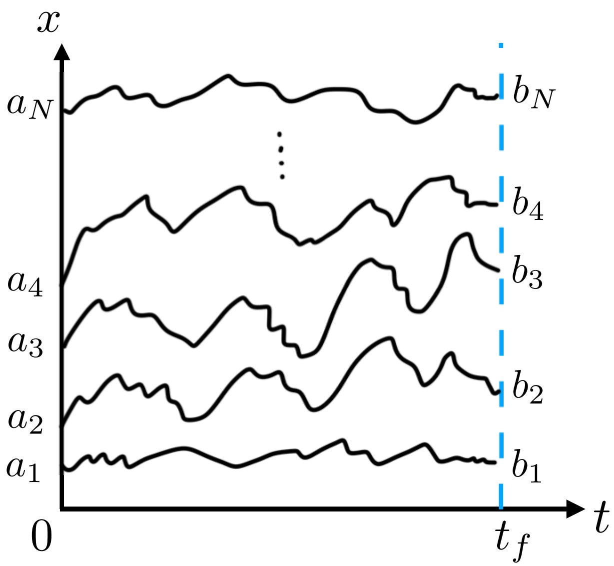


Vicious BB

↓

Dyson BB for $\beta = 2$

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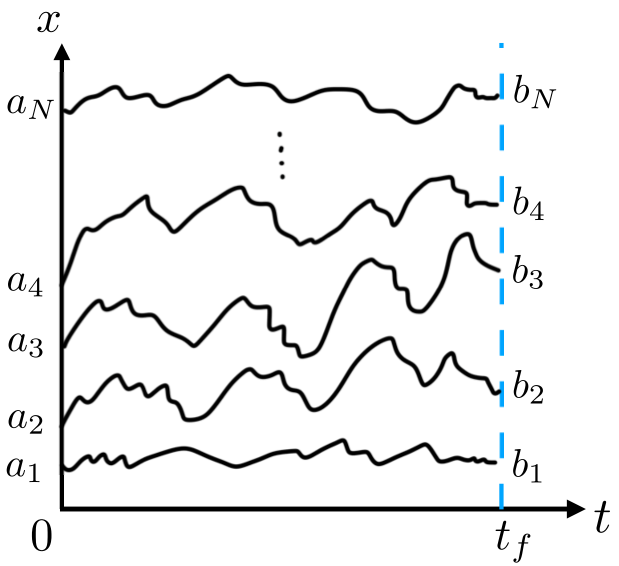


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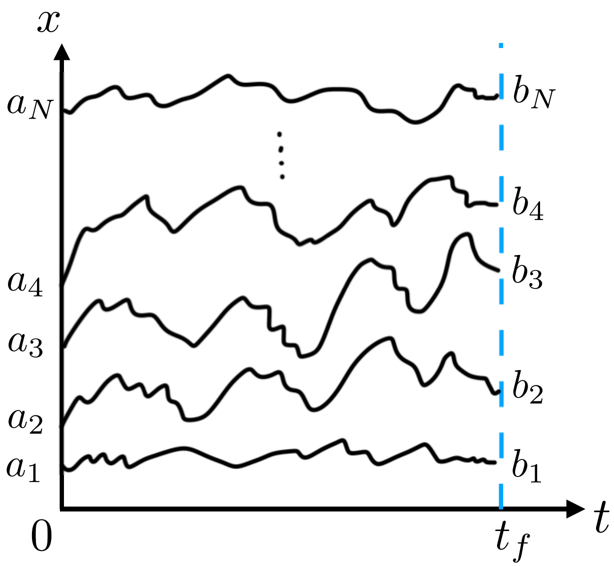


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Efficient numerical generation of VBB

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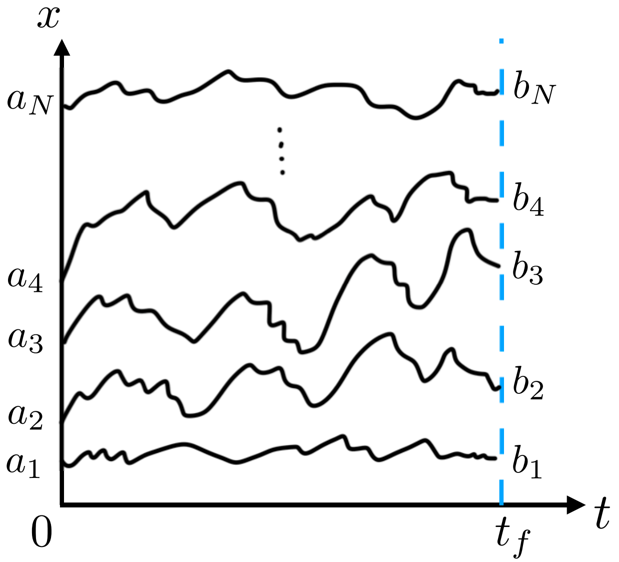


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↙ ↘

Efficient numerical generation of VBB

Exact results for the density of VBB

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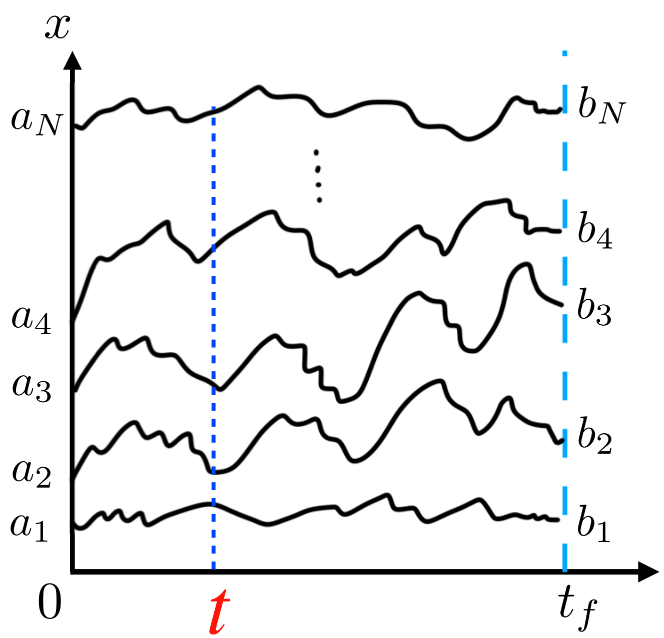


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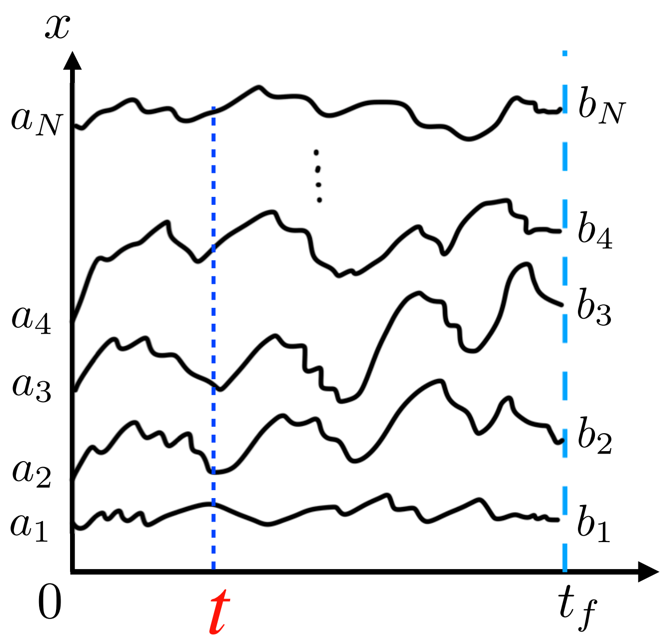


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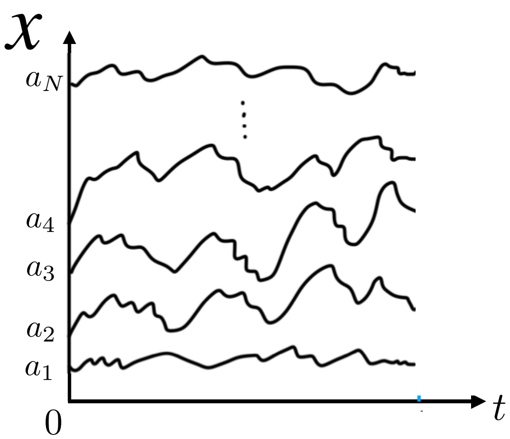
Q1: how to simulate such flat to flat configurations efficiently (i.e., beyond the costly naive way) ?

# Outline

- Mapping between Vicious Brownian Bridge and Dyson Brownian Bridge

Effective Langevin equation for Dyson Brownian Bridge

## Vicious Br. motion (VBM)



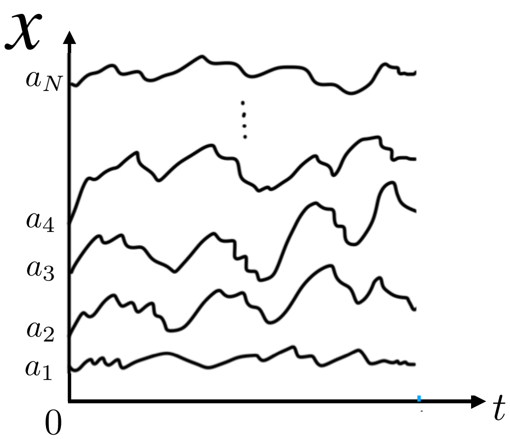

$$\frac{dx_i}{dt} = \frac{1}{\sqrt{N}} \eta_i(t)$$

$N$ indep. Gaussian white noises

conditioned on $x_1(t) \leq x_2(t) \leq \cdots \leq x_N(t)$

init. cond. $x_i(0) = a_i$

Vicious Br. motion
(VBM)




$$\frac{dx_i}{dt} = \frac{1}{\sqrt{N}}\eta_i(t)$$

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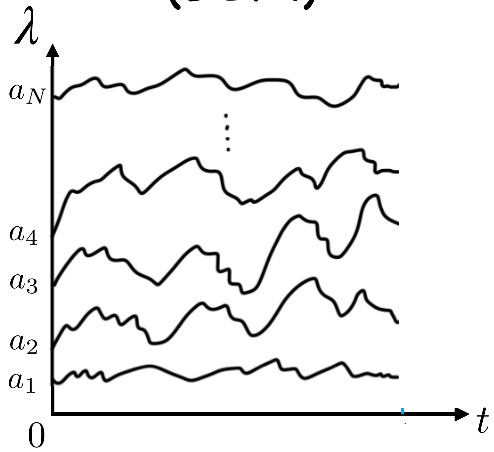
init. cond. $x_i(0) = a_i$

Dyson Br. motion
(DBM)



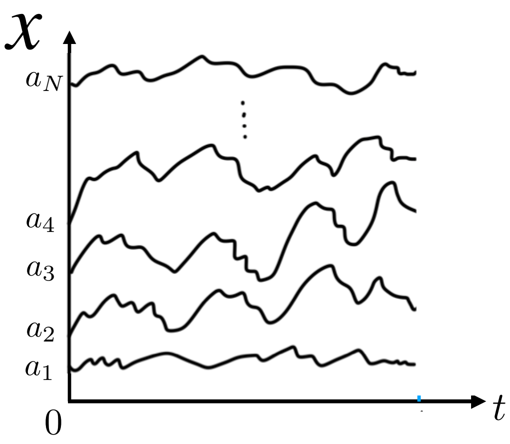

$$\frac{d\lambda_i(t)}{dt} = \frac{1}{N}\sum_{j(\neq i)=1}^{N}\frac{1}{\lambda_i(t)-\lambda_j(t)} + \sqrt{\frac{2}{\beta N}}\,\xi_i(t)$$

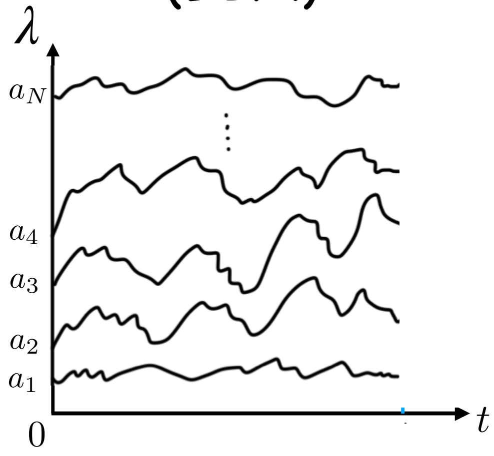
conditioned on $\lambda_1(t) \leq \lambda_2(t) \leq \cdots \leq \lambda_N(t)$

init. cond. $\lambda_i(0) = a_i$

$\beta \longrightarrow$ Dyson 's index

| Vicious Br. motion (VBM) | Dyson Br. motion (DBM) |
| --- | --- |

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init. cond. $x_i(0) = a_i$

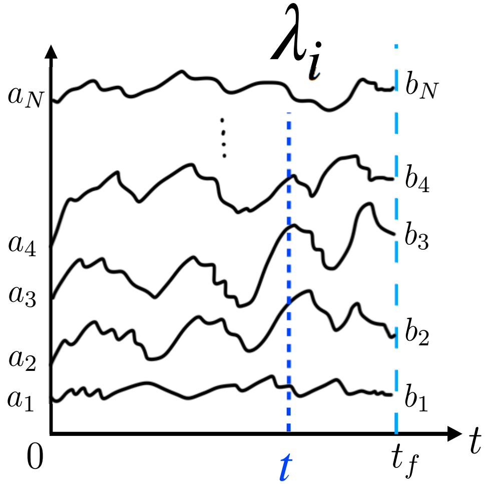
init. cond. $\lambda_i(0) = a_i$

$\beta \longrightarrow$ Dyson 's index

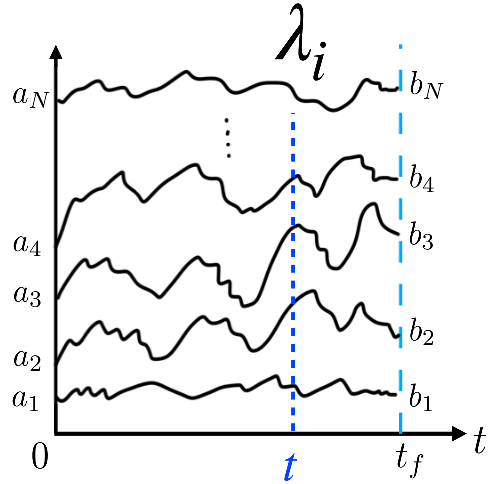
$$P_{\mathrm{VBM}}(\vec{\lambda}, t \,|\, \vec{a}, 0) = \frac{\Delta(\vec{a})}{\Delta(\vec{\lambda})} P_{\mathrm{DBM},\beta=2}(\vec{\lambda}, t \,|\, \vec{a}, 0)$$

where $\Delta(\vec{\lambda}) = \prod_{i<j}(\lambda_j - \lambda_i)$

Vicious Br. bridge
(VBB)

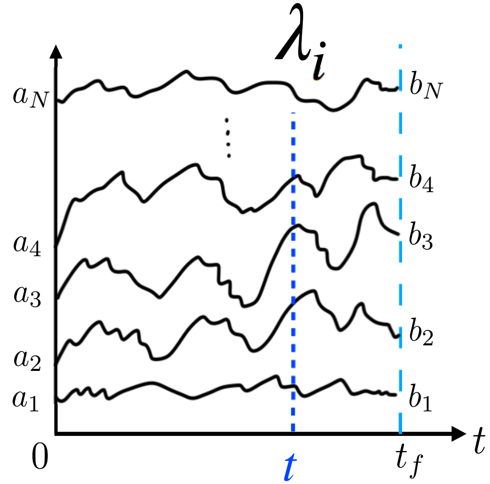



Dyson Br. bridge
(DBB)

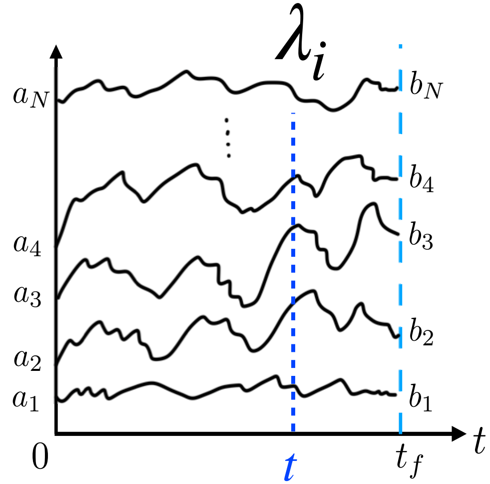



$$P_{\mathrm{VBB}}(\overrightarrow{\lambda}, t \,|\, \overrightarrow{b}, \overrightarrow{a}, t_f) =$$

Vicious Br. bridge (VBB)




Dyson Br. bridge (DBB)




$$P_{\mathrm{VBB}}(\overrightarrow{\lambda}, t \,|\, \overrightarrow{b}, \overrightarrow{a}, t_f) = \frac{P_{\mathrm{VBM}}(\overrightarrow{\lambda}, t \,|\, \overrightarrow{a}, 0)\, P_{\mathrm{VBM}}(\overrightarrow{b}, t_f \,|\, \overrightarrow{\lambda}, t)}{P_{\mathrm{VBM}}(\overrightarrow{b}, t_f \,|\, \overrightarrow{a}, 0)}$$

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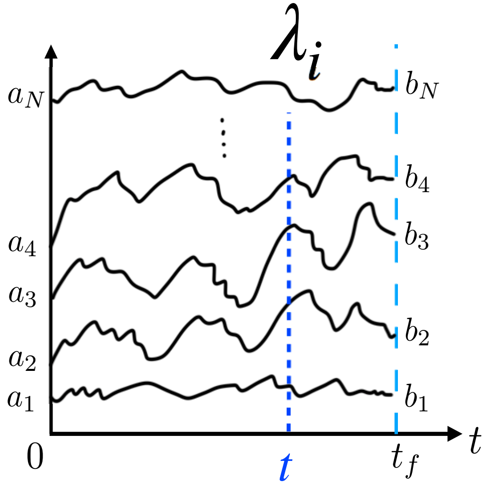


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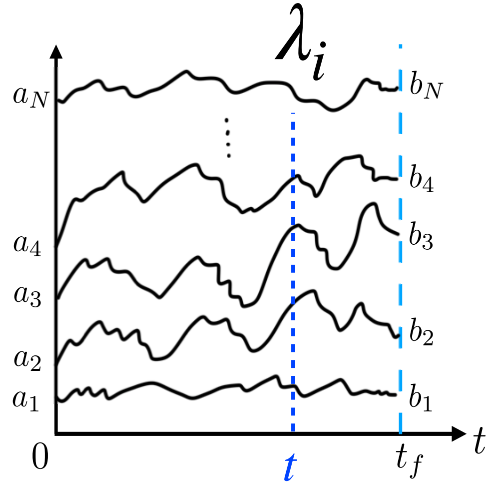


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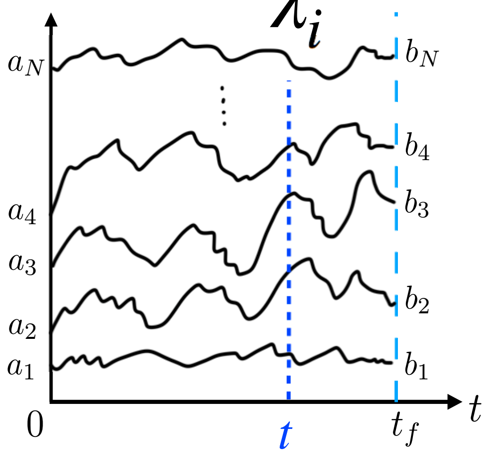
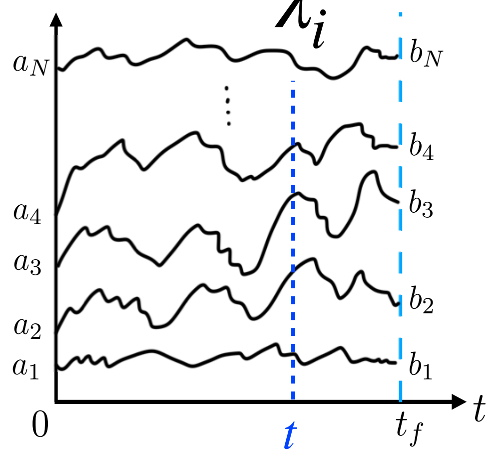
Vicious Br. bridge (VBB)




Dyson Br. bridge (DBB)




$$P_{\mathrm{VBB}}(\vec{\lambda}, t \,|\, \vec{b}, \vec{a}, t_f) = \frac{P_{\mathrm{VBM}}(\vec{\lambda}, t \,|\, \vec{a}, 0)\, P_{\mathrm{VBM}}(\vec{b}, t_f \,|\, \vec{\lambda}, t)}{P_{\mathrm{VBM}}(\vec{b}, t_f \,|\, \vec{a}, 0)}$$

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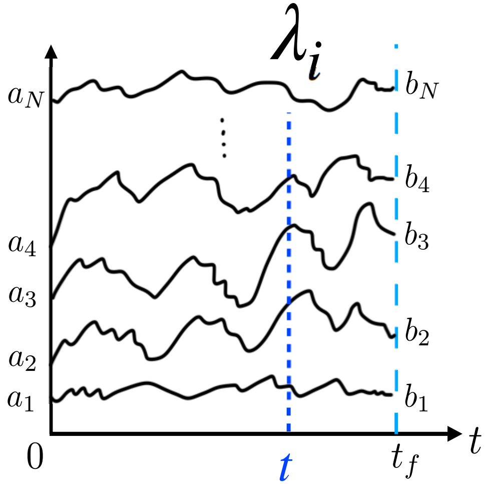
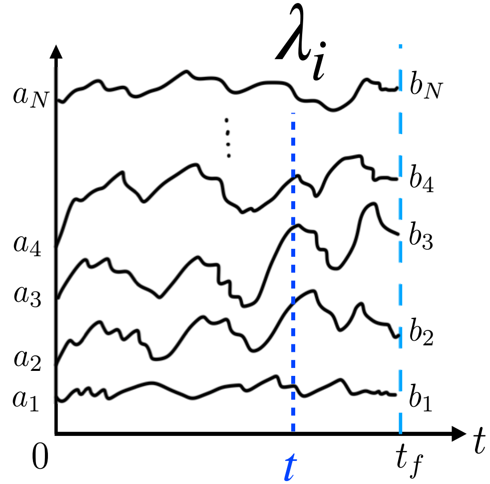
Vicious Br. bridge (VBB)




Dyson Br. bridge (DBB)

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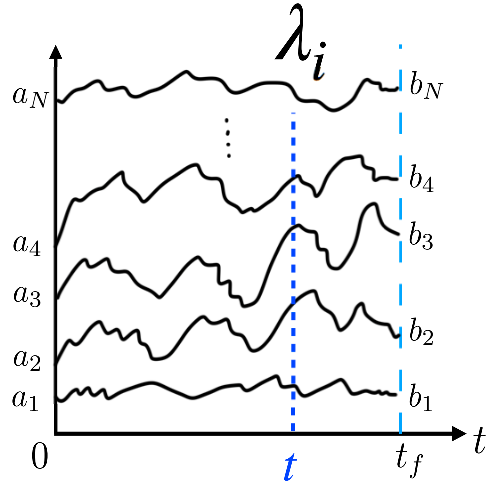
Vicious Br. bridge (VBB)

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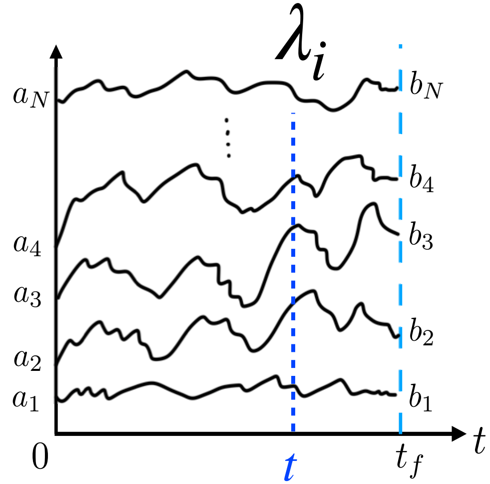


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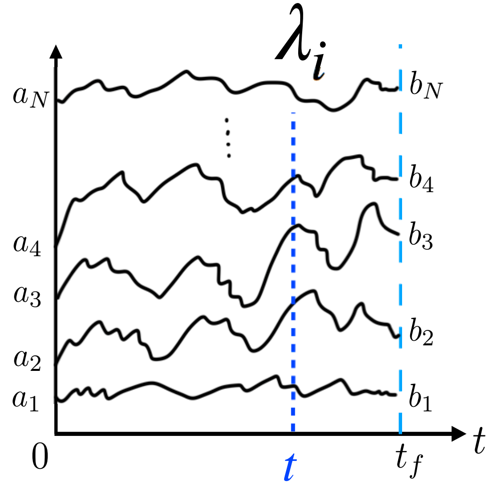
Vicious Br. bridge (VBB)

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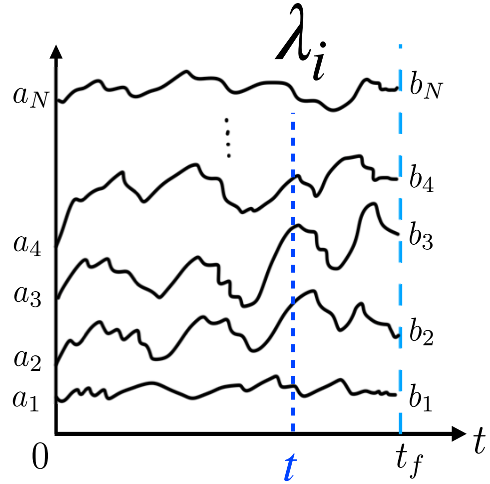


$$P_{\rm VBB}(\vec{\lambda}, t\,|\,\vec{b}, \vec{a}, t_f) = \frac{P_{\rm VBM}(\vec{\lambda}, t\,|\,\vec{a}, 0)\, P_{\rm VBM}(\vec{b}, t_f\,|\,\vec{\lambda}, t)}{P_{\rm VBM}(\vec{b}, t_f\,|\,\vec{a}, 0)}$$

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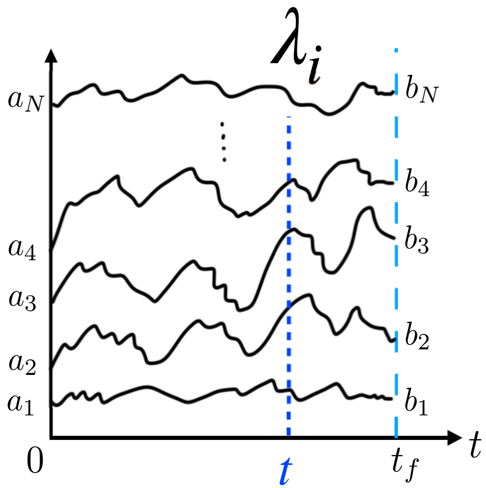
$$= P_{{\rm DBB},\beta=2}(\vec{\lambda}, t\,|\,\vec{b}, \vec{a}, t_f)$$

Vicious Br. bridge (VBB)

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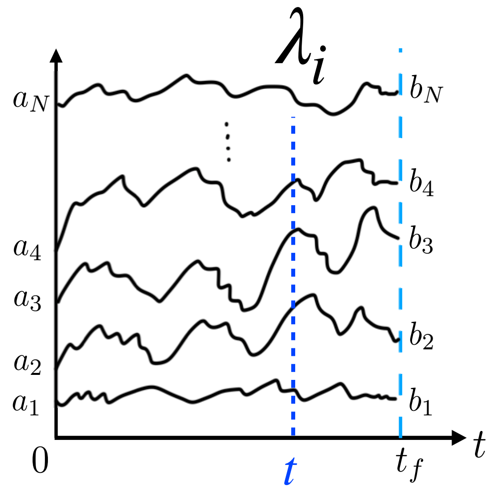


$$\Rightarrow \quad P_{\mathrm{VBB}}(\vec{\lambda}, t \,|\, \vec{b}, \vec{a}, t_f) = P_{\mathrm{DBB},\, \beta=2}(\vec{\lambda}, t \,|\, \vec{b}, \vec{a}, t_f)$$

Vicious Br. bridge
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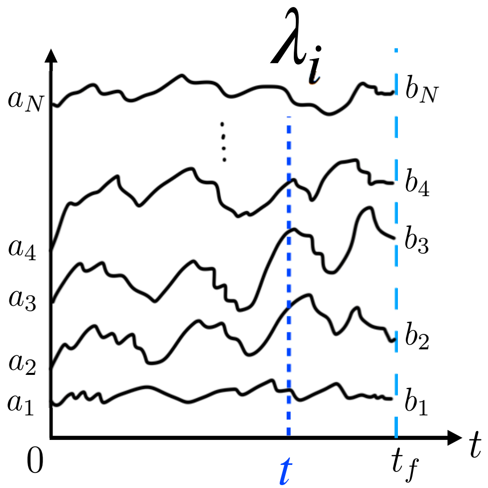


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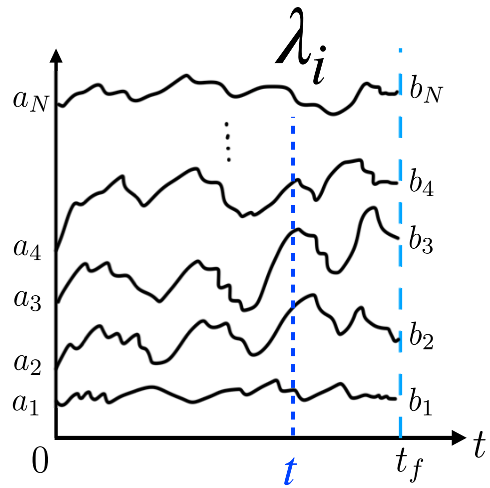


$$\Rightarrow \quad P_{\mathrm{VBB}}(\vec{\lambda}, t \,|\, \vec{b}, \vec{a}, t_f) = P_{\mathrm{DBB},\,\beta=2}(\vec{\lambda}, t \,|\, \vec{b}, \vec{a}, t_f)$$

$$\Rightarrow \quad \{\lambda_1, \lambda_2, \cdots, \lambda_N\}_{\mathrm{VBB}} \equiv \{\lambda_1, \lambda_2, \cdots, \lambda_N\}_{\mathrm{DBB},\,\beta=2}$$

Grela, S. N. M., Schehr (2021)

# Effective Langevin Eq. for DBB $(\beta = 2)$

$$\overbrace{P_{\mathrm{DBB},\beta=2}(\overrightarrow{\lambda}, t \,|\, \overrightarrow{b}, \overrightarrow{a}, t_f)}^{\tilde{P}} = \frac{\overbrace{P_{\mathrm{DBM},\beta=2}(\overrightarrow{\lambda}, t \,|\, \overrightarrow{a}, 0)}^{P}\;\overbrace{P_{\mathrm{DBM},\beta=2}(\overrightarrow{b}, t_f \,|\, \overrightarrow{\lambda}, t)}^{Q}}{P_{\mathrm{DBM},\beta=2}(\overrightarrow{b}, t_f \,|\, \overrightarrow{a}, 0)}$$

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- Write explicit Fokker-Planck Eq. for $P$ and $Q$ separately

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- For the flat final config. $b_i = \dfrac{i-1}{N} \quad , \quad i = 1, 2, \cdots, N$

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This FP Eq. for $\tilde{P}$ simplifies

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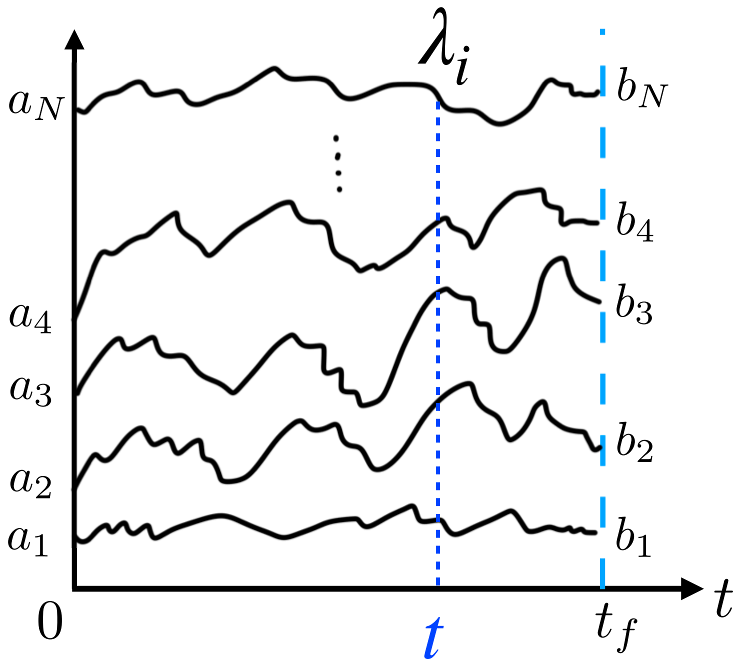
This FP Eq. for $\tilde{P}$ simplifies

⇨ One can read off the effective Langevin Eq. associated to $\tilde{P}$

# Effective Langevin Eq. for DBB $(\beta = 2)$




Flat final config. $b_i = \dfrac{i-1}{N}$

Grela, S. N. M., Schehr (2021)

$$\frac{d\lambda_i}{dt} = -\frac{\lambda_i}{t_f - t} + \frac{1}{N(t_f - t)} \sum_{j(\neq i)} \frac{e^{\frac{\lambda_i}{t_f - t}}}{e^{\frac{\lambda_i}{t_f - t}} - e^{\frac{\lambda_j}{t_f - t}}} + \frac{1}{\sqrt{N}} \xi_i(t) \qquad i = 1, \cdots, N$$

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valid for any initial condition $\overrightarrow{\lambda}(t=0) = \overrightarrow{a}$

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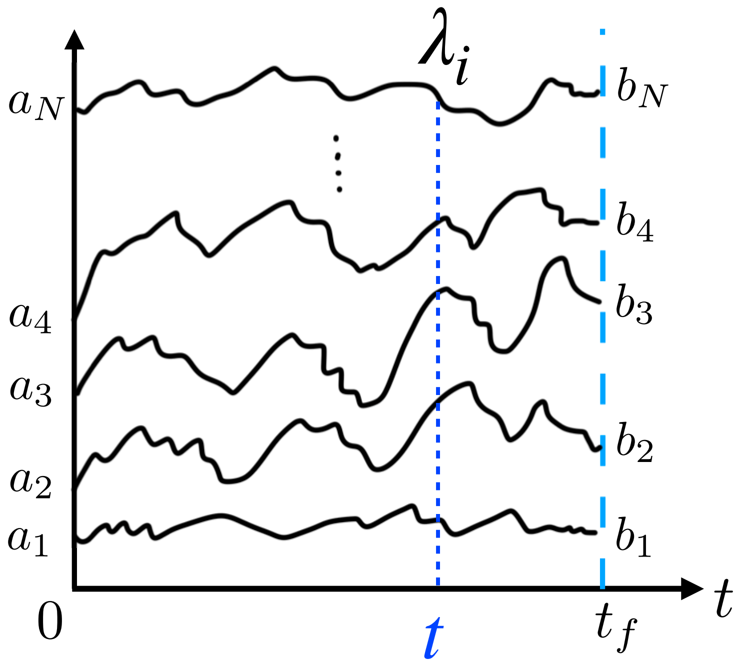


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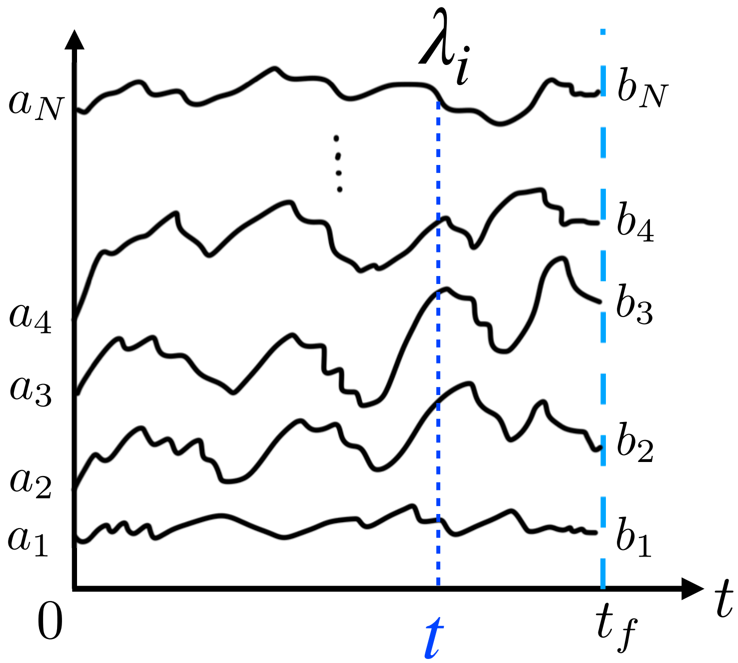
$N$ indep. Gaussian white noises

valid for any initial condition $\vec{\lambda}(t=0) = \vec{a}$

it automatically ensures (i) final flat config.

(ii) non-crossing during $[0, t_f]$

# Effective Langevin Eq. for DBB $(\beta = 2)$

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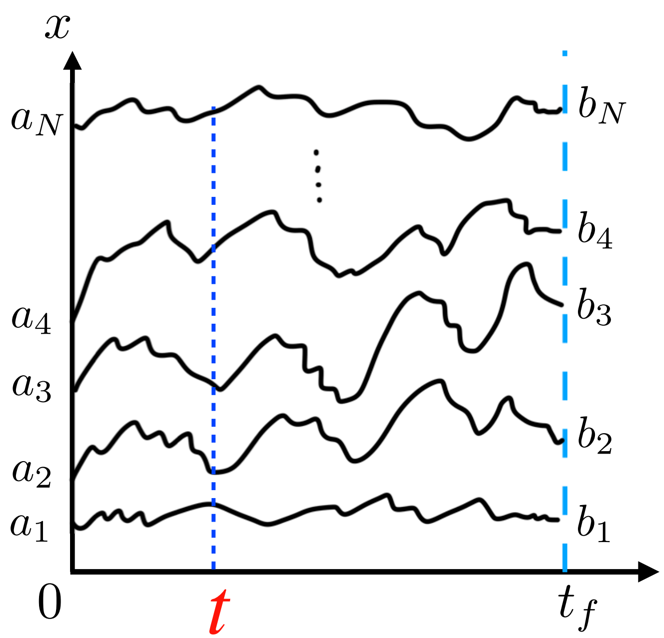
$N$ indep. Gaussian white noises

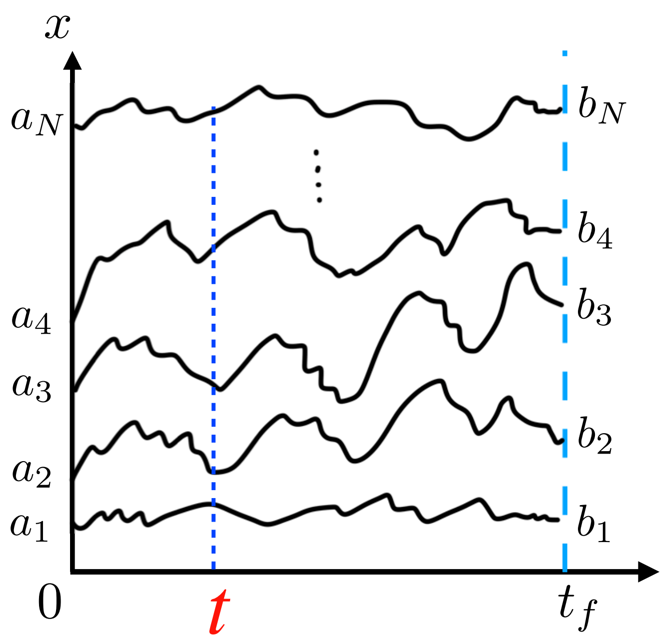
Discretizing in time $\Longrightarrow$ generates VBB trajectories in a flat to flat geometry in a rejection free way

# Nonintersecting (vicious) Brownian bridges




for $a_i = b_i = \dfrac{i-1}{N}, i = 1, 2, \cdots, N$

# Nonintersecting (vicious) Brownian bridges



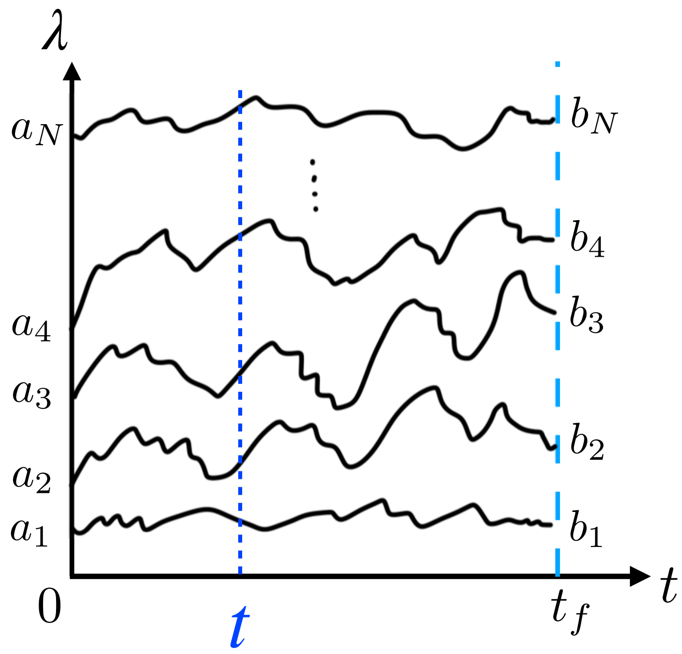

for $a_i = b_i = \dfrac{i-1}{N}, i = 1,2,\cdots,N$

Q2: what is the average density at some intermediate time $0 \le t \le t_f$ ?

# Joint distribution of the positions of the VBB

- Vicious Brownian Bridges in flat-to-flat geometry

$$P_{\rm VBB}(\overrightarrow{\lambda}, t \,|\, \overrightarrow{b}, \overrightarrow{a}, t_f) = \frac{P_{\rm VBM}(\overrightarrow{\lambda}, t \,|\, \overrightarrow{a}, 0)\, P_{\rm VBM}(\overrightarrow{b}, t_f \,|\, \overrightarrow{\lambda}, t)}{P_{\rm VBM}(\overrightarrow{b}, t_f \,|\, \overrightarrow{a}, 0)}$$



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- Vicious Brownian Bridges in flat-to-flat geometry

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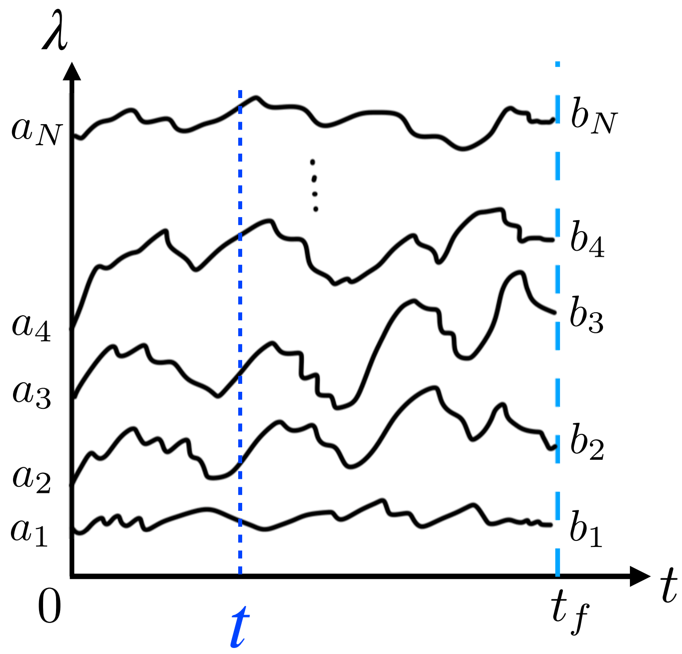


using Karlin-Mc Gregor formula

for $a_i = b_i = \dfrac{i-1}{N}, i = 1, 2, \cdots, N$

$$P_{\rm VBB}\left(\vec{\lambda}, t \,|\, \vec{a}, 0; \vec{b}, t_f\right) \propto \det_{1 \le i,j \le N}\left(e^{-\frac{N}{2t}\left(\lambda_i - a_j\right)^2}\right) \det_{1 \le i,j \le N}\left(e^{-\frac{N}{2(t_f - t)}\left(b_i - \lambda_j\right)^2}\right)$$

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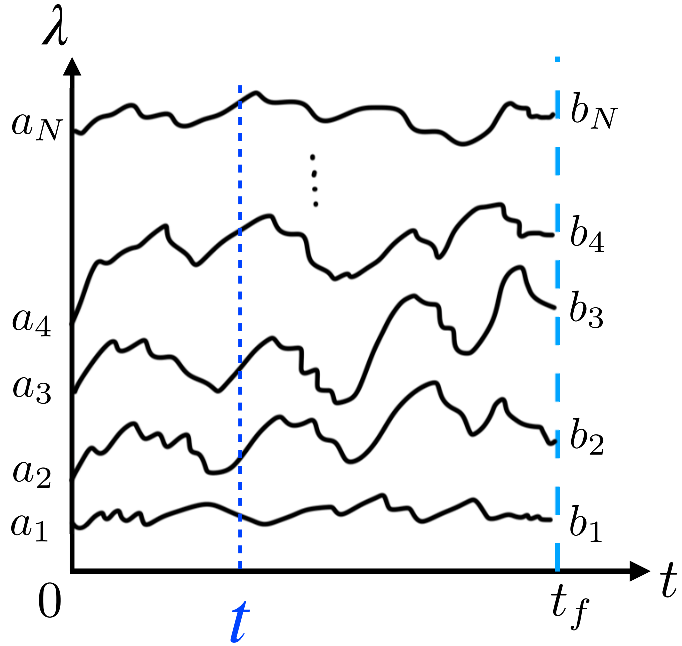


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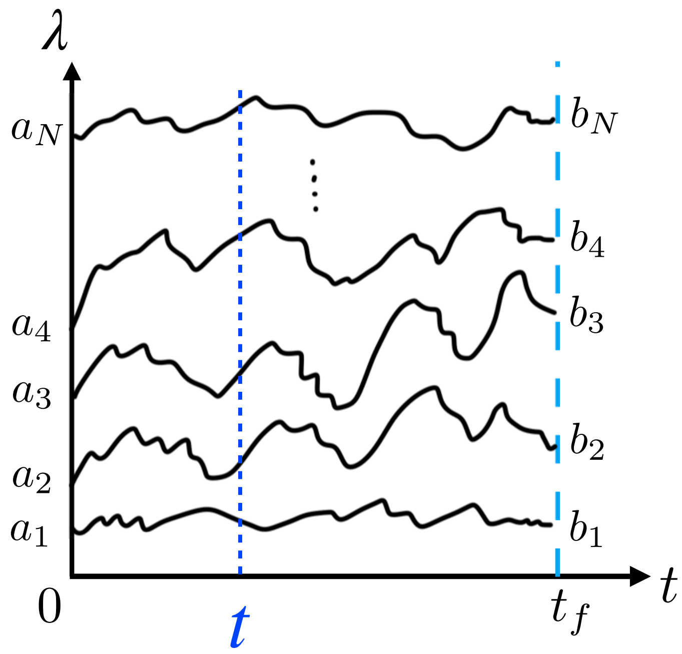


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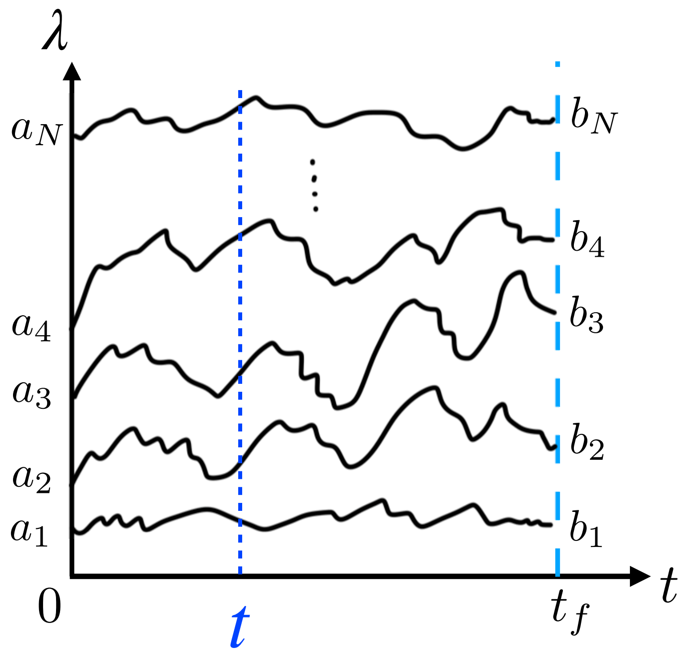


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⇨ Chern-Simons model (Mariño 2005), bi-orthogonal Stieltjes-Wigert polynomials (Dolivet & Tierz 2007, Katori & Takahashi 2012)

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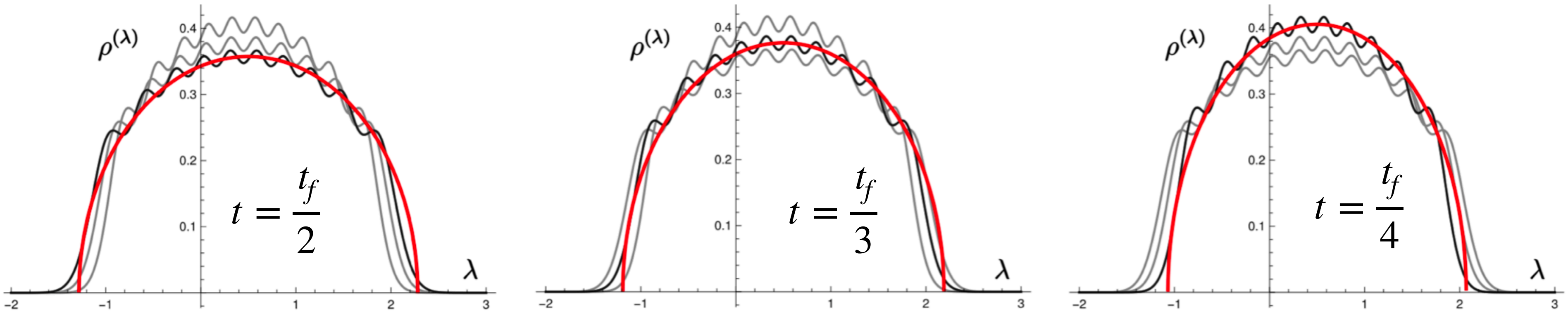

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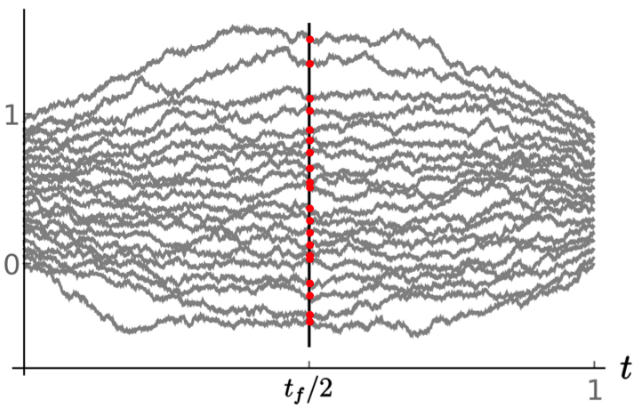


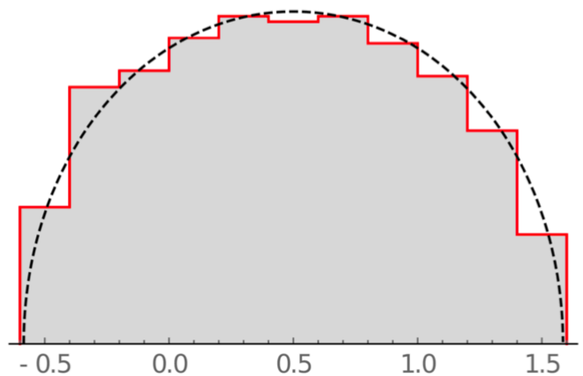


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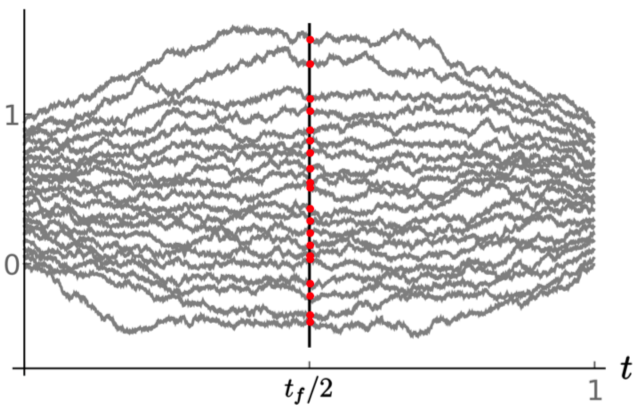


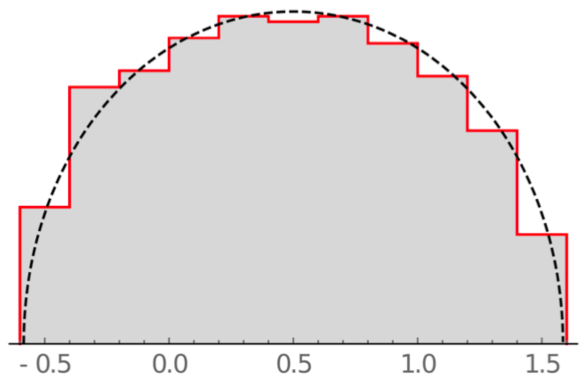


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M. Marino (2005), P. J. Forrester (2021)

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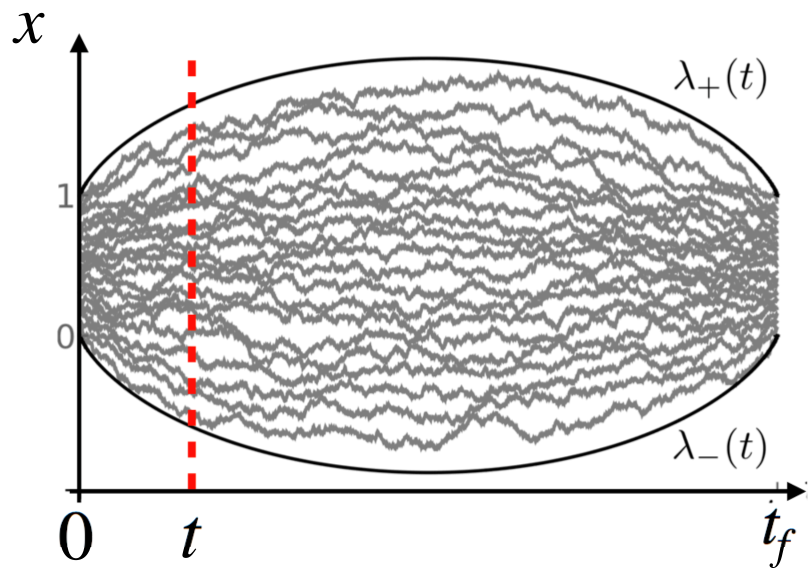


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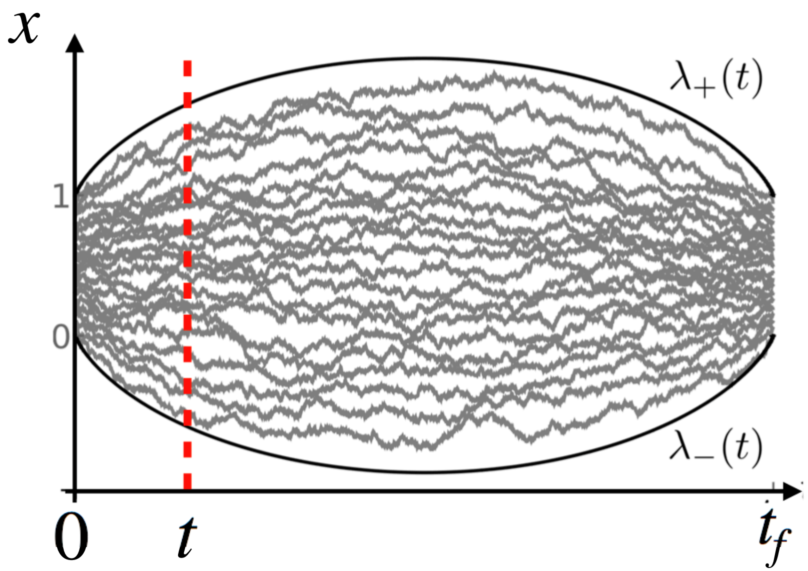


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- However, our effective Langevin equation gives access to this average density, in principle for any $t \in [0,t_f]$

# Towards computing the average density from the effective Langevin equation

- For the equi-spaced/flat final condition: $b_i = \dfrac{i-1}{N}, i = 1,2,\cdots,N$

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⇨ multiplicative noise (Ito prescription)

# Average particle density via Burgers' equation

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⇒

Can one derive an evolution equation (i.e., a PDE) for $G_N$ ?

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- This is different from the « standard » Burgers' equation for DBM

Blaizot, Nowak (2010), Allez, Bouchaud, Guionnet (2012),
Blaizot, Grela, Nowak, Warchol (2015), Krajenbrinck, Le Doussal, O'Connel (2020)

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$$H(y;\theta) = e^{-G(y;\theta)} \quad , \quad \text{where} \quad X = e^y \quad \text{and} \quad T = e^{-1/t_f}$$

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- Limiting density: $\rho(X;\theta) = \lim_{N\to\infty} \rho_N(X;\theta) = \frac{t_f}{\pi} \lim_{\epsilon\to 0_+} \mathrm{Im}\left[\frac{1}{y} G(\ln y;\theta)\right]_{y=X-i\epsilon}$
- From $\rho(X;\theta)$ we can compute the average density in the original $(\lambda, t)$ coordinates

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Grela, S. N. M., Schehr (2021)

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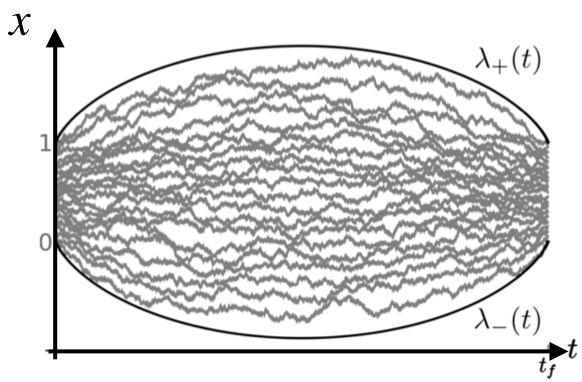


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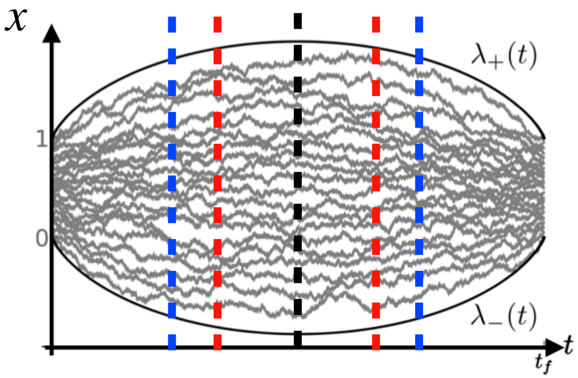


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- By solving $H^\theta = \dfrac{1}{X}\dfrac{1 - H}{T - H}$ , $\theta = \dfrac{t}{t_f - t}$ , for $\theta = 2,3,4$ we obtain the density explicitly for $t = \dfrac{t_f}{4}, \dfrac{t_f}{3}, \dfrac{t_f}{2}, \dfrac{2t_f}{3}, \dfrac{3t_f}{4}$

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- Connections to other models:
  - Chern-Simons model
  - Bi-orthogonal Stieltjes-Wigert polynom.
  - Muttalib-Borodin ensemble
  - Harish Chandra-Itzykson-Zuber integrals